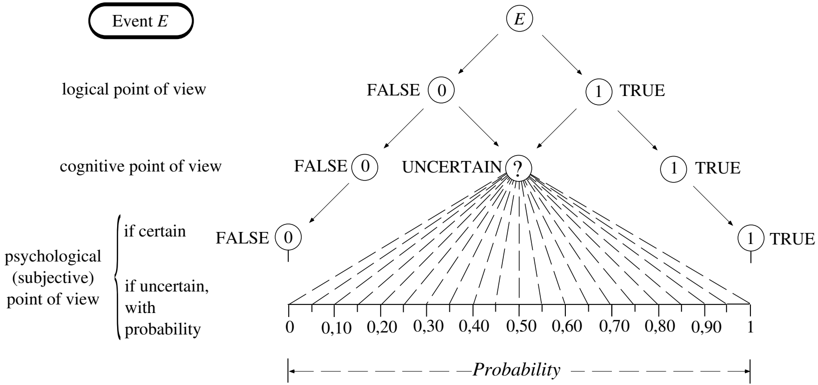
Measurements, uncertainties and probabilistic inference/forecasting

Giulio D'Agostini

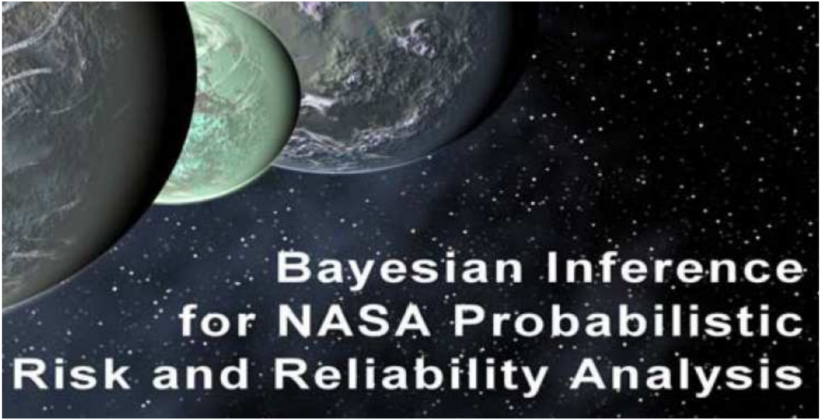
Università di Roma La Sapienza e INFN
Roma, Italy

False, True and probable



An helpful diagram

The previous diagram seems to help the understanding of the concept of probability



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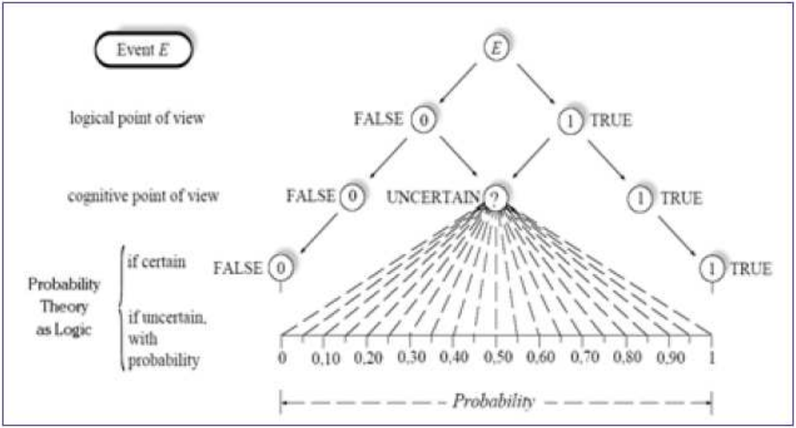

• Figure 2-1. Graphical abstraction of probability as a measure of information (adapted from "Probability and Measurement Uncertainty in Physics" by D'Agostini, [1995]).

(. . . but NASA guys are afraid of 'subjective', or 'psychological')

Beliefs and ‘coherent’ bets

Remarks

- Subjective does not mean arbitrary!

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“His [Bouvard] calculations give him the mass of Saturn as 3,512th part of that of the sun. Applying my probabilistic formulae to these observations, I find that the odds are 11,000 to 1 that the error in this result is not a hundredth of its value.” (Laplace)

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Is a 'conventional' 95% C.L. lower/upper bound a 19 to 1 bet?
It does not imply one has to be 95% confident on something!
If you think so, then you could make a bad bet!

A clear statement from a honest expert

MODIFIED FREQUENTIST ANALYSIS OF SEARCH RESULTS (THE CL_s METHOD)

A. L. Read
University of Oslo, Department of Physics, P.O. Box 1048, Blindern, 0316 Oslo 3, Norway

CONCLUSION

It neither adheres to the frequentist principle of coverage (it overcovers *by design* as the experimental sensitivity to the hypothetical signal vanishes) nor does it indicate the bounds of a Bayesian subjective probability distribution.

Let me try to make an important point about the previous sentence as clearly and simply as possible (even my friends claim I got it wrong all the three times I tried to explain this in my presentation): The lower bounds on the Higgs mass that are quoted for the direct Higgs searches at LEP say *absolutely nothing* about the *probability of the Higgs mass* being higher or lower than some value.

To make such a statement the direct search results must be first folded with a prior probability distribution for the Higgs mass [15].

Beliefs and bet

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'Would you bet a hundred dollars a dollar that parity is not violated?' he asked.

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> *"Although I felt sure that parity would not be violated, there was a possibility that it would be, and it was important to find out.*
> *'Would you bet a hundred dollars a dollar that parity is not violated?' he asked.*
> *'No. But fifty dollars I will'"*

(Feynman)

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Standard textbook definitions

$$p = \frac{\#\,\text{favorable cases}}{\#\,\text{possible equiprobable cases}}$$

$$p = \frac{\#\,\text{times the event has occurred}}{\#\,\text{independent trials under same conditions}}$$

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Note!: *"lorsque rien ne porte à croire que l'un de ces cas doit arriver plutot que les autres" (Laplace)*

Replacing 'equi-probable' by 'equi-possible' is just cheating students (as I did in my first lecture on the subject. . .).

Standard textbook definitions

It is easy to check that ‘scientific’ definitions suffer of circularity, plus other problems

$$p = \frac{\#\,\text{favorable cases}}{\#\,\text{possible {\color{red}equiprobable} cases}}$$

$$p = \lim_{n\to\infty} \frac{\#\,\text{times the event has occurred}}{\#\,\text{{\color{red}independent} trials under {\color{red}same condition}}}$$

Future ⇔ Past (belief!)

$n \to \infty$: → *“usque tandem?”*

→ *“in the long run we are all dead”*

→ It limits the range of applications

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Future ⇔ Past: avoid the end of the *inductivist turkey*!

‘Definitions’ → evaluation rules

Very useful evaluation rules

$$A) \quad p = \frac{\#\,\text{favorable cases}}{\#\,\text{possible equiprobable cases}}$$

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If the implicit beliefs are well suited for each case of application.

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BUT they cannot define the concept of probability!

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In the probabilistic approach we are following

- Rule A is recovered immediately (under the assumption of equiprobability, when it applies).
- Rule B results from a theorem of Probability Theory (under well defined assumptions).

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- Rule B results from a theorem of Probability Theory (under well defined assumptions):
  $\Rightarrow$ Laplace’s rule of succession (see later)

Mathematics of beliefs

The good news:

> *The basic laws of degrees of belief are the same we get from the inventory of favorable and possible cases, or from events occurred in the past.*

It can be proved that

> *the requirement of* ***coherence*** *leads to the famous 4 basic rules* $\Longrightarrow$

[Details skipped. . .]

Basic rules of probability

1. $0 \leq P(A \,|\, I) \leq 1$
2. $P(\Omega \,|\, I) = 1$
3. $P(A \cup B \,|\, I) = P(A \,|\, I) + P(B \,|\, I) \quad [\text{if } P(A \cap B \,|\, I) = \emptyset]$
4. $P(A \cap B \,|\, I) = P(A \,|\, B, I) \cdot P(B \,|\, I) = P(B \,|\, A, I) \cdot P(A \,|\, I)$

Remember that probability is always conditional probability!

I is the background condition (related to information 'I_s')

→ usually implicit (we only care about 're-conditioning')

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Note: 4. does not define conditional probability.
(Probability is always conditional probability!)

Mathematics of beliefs

An even <u>better news</u>:

The fourth basic rule
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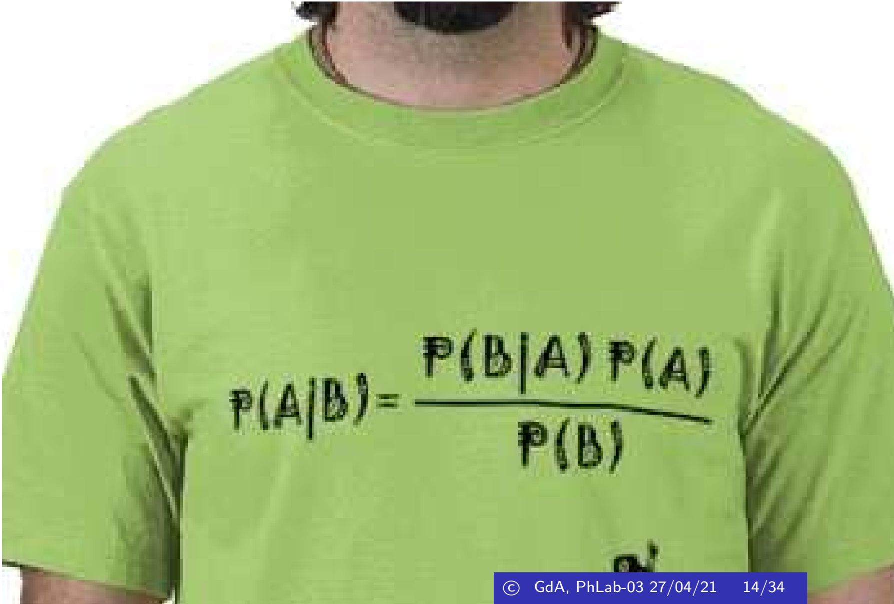
(Liberated by a curious ideology that forbids its use)

A simple, powerful formula



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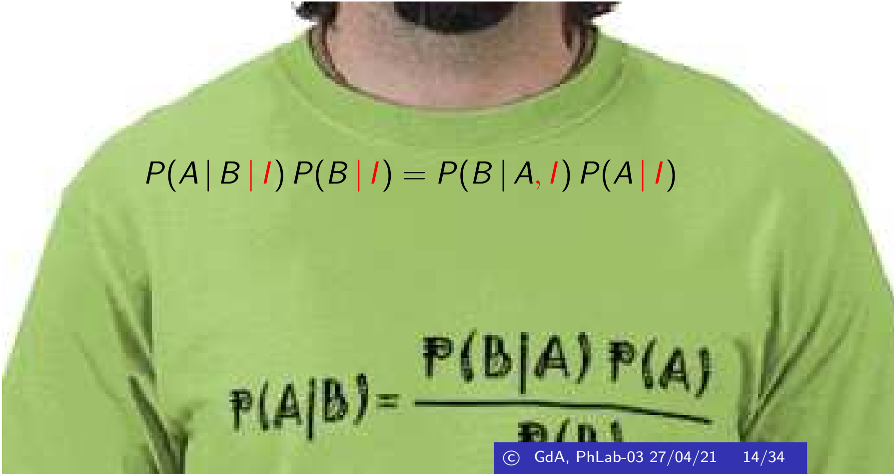

$$P(A \mid B \mid I)\, P(B \mid I) = P(B \mid A, I)\, P(A \mid I)$$

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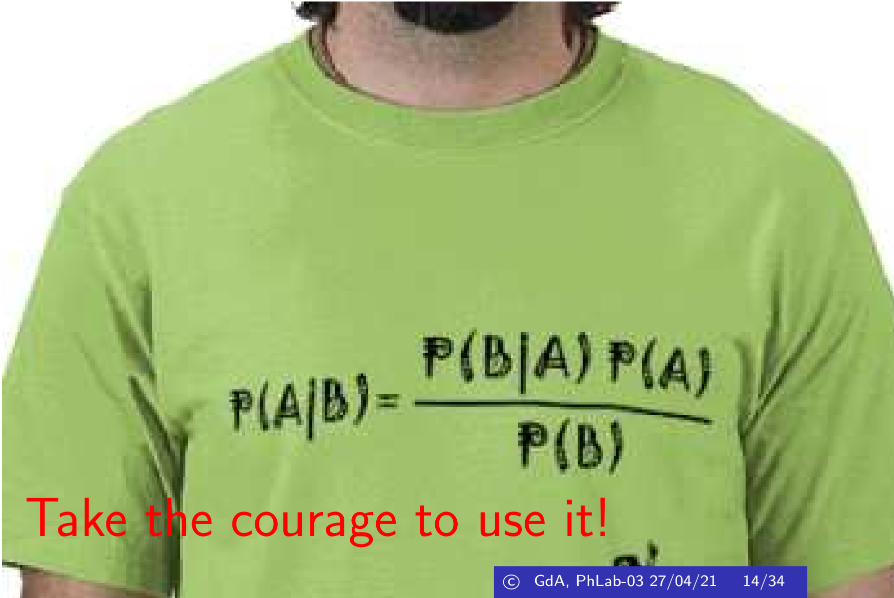

Take the courage to use it!

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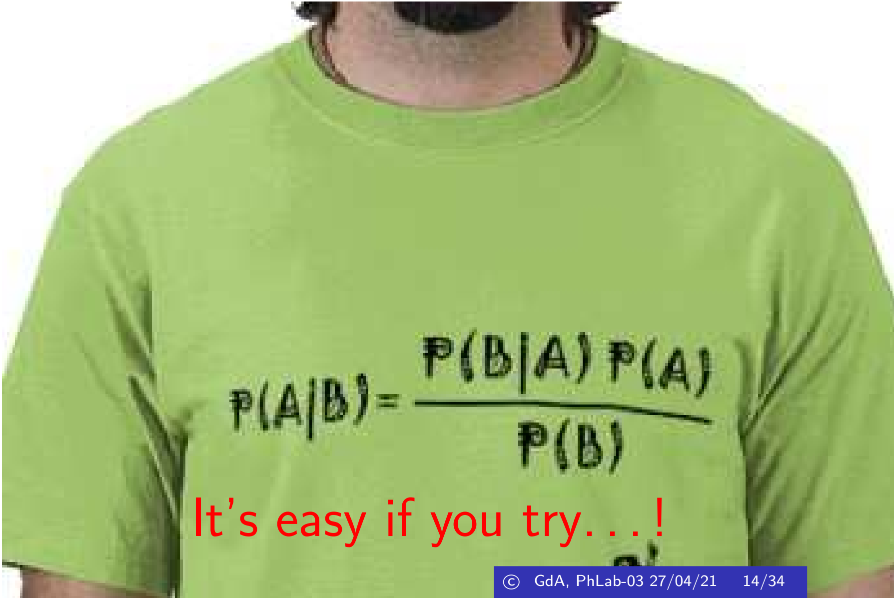
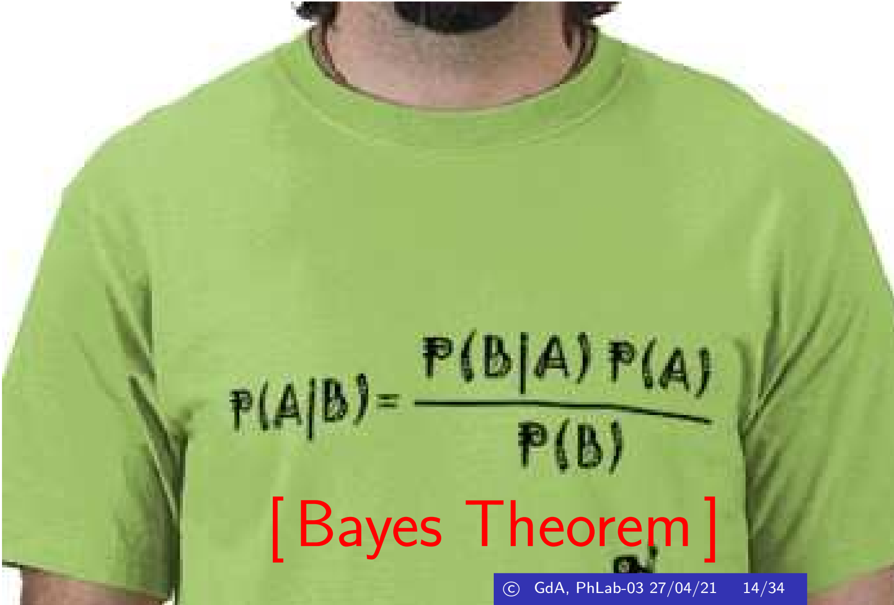


It's easy if you try. . . !

A simple, powerful formula



[Bayes Theorem]

Laplace's "Bayes Theorem"

"The greater the probability of an observed event given any one of a number of causes to which that event may be attributed, the greater the likelihood of that cause {given that event}.

$$P(C_i \,|\, E) \propto P(E \,|\, C_i)$$

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"The greater the probability of an observed event given any one of a number of causes to which that event may be attributed, the greater the likelihood of that cause {given that event}. The probability of the existence of any one of these causes {given the event} is thus a fraction whose numerator is the probability of the event given the cause, and whose denominator is the sum of similar probabilities, summed over all causes.

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$$P(C_i \,|\, E) = \frac{P(E \,|\, C_i)\, P(C_i)}{P(E)}$$

(*Philosophical Essai on Probabilities*)

[In general $P(E) = \sum_j P(E \,|\, C_j)\, P(C_j)$ (*weighted average*, with weigths being the probabilities of the conditions) if C_j form a *complete class of hypotheses*]

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$$P(C_i \mid E) \quad = \quad \frac{P(E \mid C_i)\, P(C_i)}{P(E)} = \frac{P(E \mid C_i)\, P(C_i)}{\sum_j P(E \mid C_j)\, P(C_j)}$$

"This is the fundamental principle $^{(*)}$ of that branch of the analysis of chance that consists of reasoning a posteriori from events to causes"

(*) In his "Philosophical essay" Laplace calls 'principles' the 'fundamental rules'.

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Most convenient way to remember Bayes theorem

Laplace's teaching

$$\frac{P(H_0 \,|\, \text{data})}{P(H_1 \,|\, \text{data})} = \frac{P(\text{data} \,|\, H_0)}{P(\text{data} \,|\, H_1)} \times \frac{P(H_0)}{P(H_1)}$$

- We should possibly use the data, rather then the test variables 'θ' (χ^2 etc);

  [although in some case 'sufficient summaries' do exist]

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- If $P(\text{data} \mid H_i) = 0$, it follows $P(H_i \mid \text{data}) = 0$:
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  $\Rightarrow$ falsification (the 'serious' one) is a corollary of the theorem, rather than a principle.
- There is no conceptual problem with the fact that $P(\text{data}\,|\,H_1) \to 0$ (e.g. 10^{-37}), provided the ratio $P(\text{data}\,|\,H_0)/P(\text{data}\,|\,H_1)$ is not undefined.

Bayes factor ('likelihood ratio')

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$$\text{Prob. ratio}\big|_{\textit{posterior}} \;=\; \text{Bayes factor} \times \text{Prob. ratio}\big|_{\textit{prior}}$$

(*prior*/*posterior* w.r.t. data)

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$$\text{Prob. ratio}\big|_{posterior} \;=\; \text{Bayes factor} \times \text{Prob. ratio}\big|_{prior}$$

(*prior*/*posterior* w.r.t. data)

If H_0 and H_1 are 'complementary', that is $H_1 = \overline{H}_0$, then

$$\textbf{posterior odds} \;=\; \textbf{Bayes factor} \times \textbf{prior odds}$$

But statistical tests do work!

Someone would object that p-values and, in general, ‘hypothesis tests’ *usually* do work!

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- Certainly! I agree!
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- But now we are also able to explain the reason.

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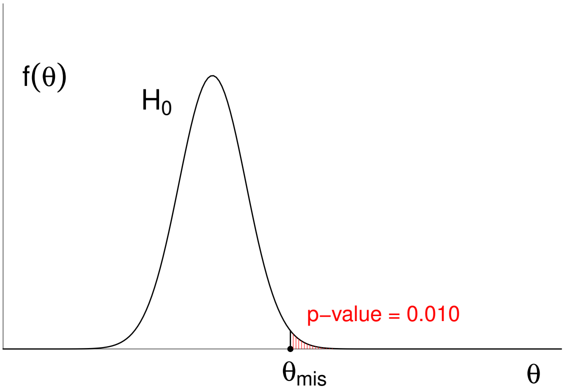
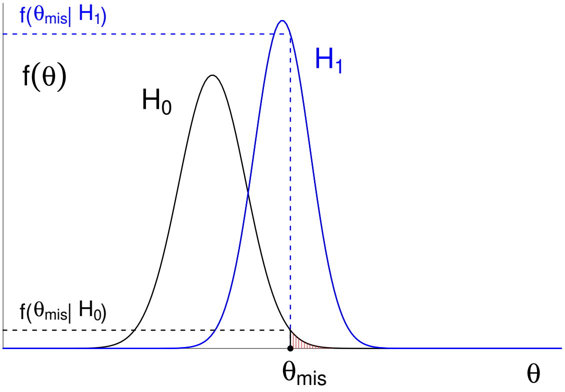


Why should the observation of θ_{mis} should diminish our confidence on H_0?

But statistical tests do work!



Because *often* we give *some chance* to a possible alternative hypothesis H_1, even if we are not able to exactly formulate it.

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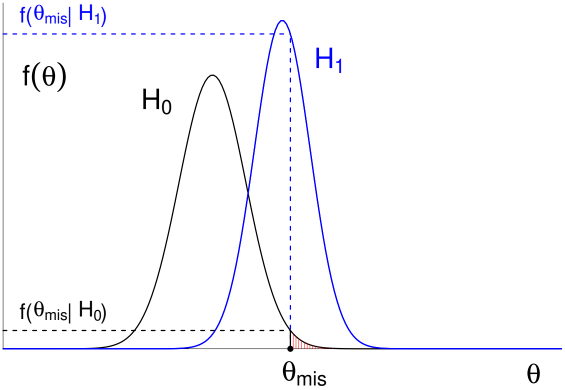

Indeed, what really matters is not the area to the right of θ_{mis}.
What matters is the ratio of $f(\theta_{mis} \,|\, H_1)$ to $f(\theta_{mis} \,|\, H_0)$!
$\Rightarrow$ to a 'small' area it corresponds a 'small' $f(\theta_{mis} \,|\, H_0)$.

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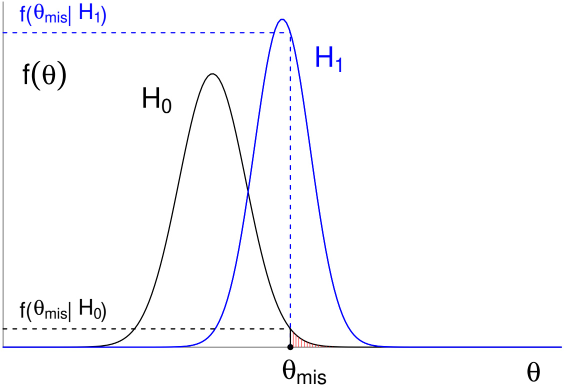

But is the alternative hypothesis H_1 is unconceivable, or hardly believable, the 'smallness' of the area is irrelevant

Telling it with Gauss' words

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A quote from the Princeps Mathematicorum (Prince of Mathematicians) is a must.

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"post illa observationes" *"ante illa observationes"*

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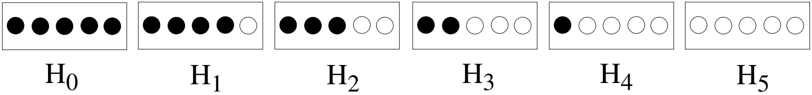
Arguments used to derive Gaussian distribution

- $f(\mu\,|\,\{x\}) \propto f(\{x\}\,|\,\mu)\cdot f_0(\mu)$
- $f_0(\mu)$ 'flat' (all values a priory equally possible)
- posterior maximized at $\mu = \overline{x}$

Application to the six box problem



Remind:

- E_1 = White
- E_2 = Black

Collecting the pieces of information we need

Our tool:

$$\boxed{P(H_j \mid E_i, I) = \frac{P(E_i \mid H_j, I)}{P(E_i \mid I)}\, P(H_j \mid I)}$$

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Our prior belief about H_j

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Probability of E_i under a well defined hypothesis H_j
It corresponds to the 'response of the apparatus' in measurements.
$\rightarrow$ likelihood (traditional, rather confusing name!)

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'decomposition law': $P(E_i \,|\, I) = \sum_j P(E_i \,|\, H_j, I) \cdot P(H_j \,|\, I)$
(→ Easy to check that it gives $P(E_i \,|\, I) = 1/2$ in our case).

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We are ready!

$\longrightarrow$ Let's play with our toy

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Now that we have set up our formalism, let's play a little

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Let's play!

- Hugin Expert (Lite – demo version);
- R scripts

Playing with the six boxes

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- History of $P(H_j \,|\, \text{obs. sequence})$.

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NO!

- Don't even think: frequentists refuse to assign probabilities to hypotheses (in general), to causes, to true values, etc. (And you have seen the results. . .)

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Simple case (no reporter/composition/etc. complications)

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(We shall see it later in the case of *continuous distributions*)

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f(p\,|\,x,n) &\propto p^{x}(1-p)^{(n-x)} \qquad [x = \#S]
\end{aligned}
$$

Inferring 'proportions'

Let's turn the toy experiment to a 'serious' physics case:

- Inferring H_j is the same as inferring the proportion of white balls:

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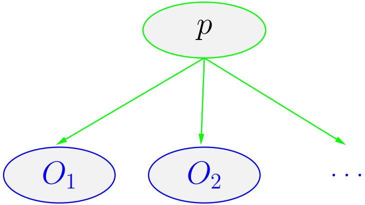
- Generalize White/Black $\longrightarrow$ Success/Failure

$\Rightarrow$ efficiencies, branching ratios, . . .

Inferring Bernoulli's trial parameter p

Making several independent trials *assuming* the same p



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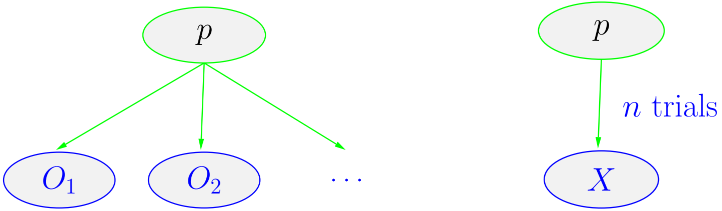

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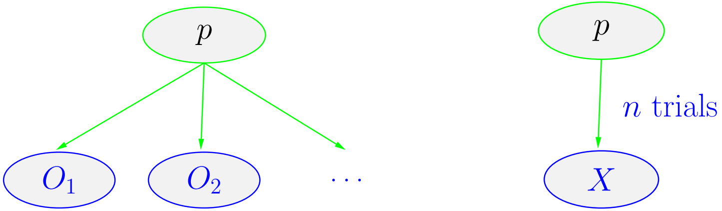

"independent Bernoulli trials" "binomial distribution"

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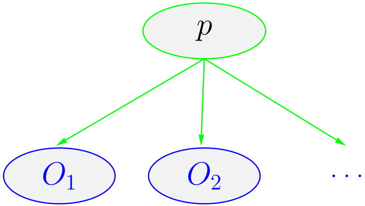

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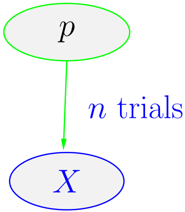
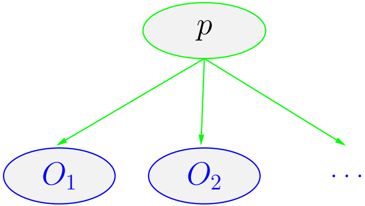


"binomial distribution"

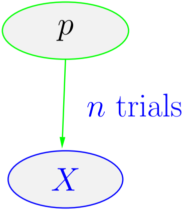
⇒ In the light of the experimental information
there will be values of p we shall believe more,
and others we shall believe less.

Inferring Bernoulli's trial parameter p

Making several independent trials *assuming* the same p



"independent Bernoulli trials"



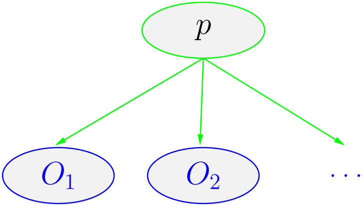
"binomial distribution"

$P(p_i \,|\, O_1, O_2, \ldots)$
$f(p \,|\, O_1, O_2, \ldots)$

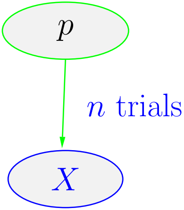
$P(p_i \,|\, X, n)$
$f(p \,|\, X, n)$

Inferring Bernoulli's trial parameter p

Making several independent trials *assuming* the same p



"independent Bernoulli trials"



"binomial distribution"

$$P(p_i \,|\, O_1, O_2, \ldots)$$
$$f(p \,|\, O_1, O_2, \ldots)$$

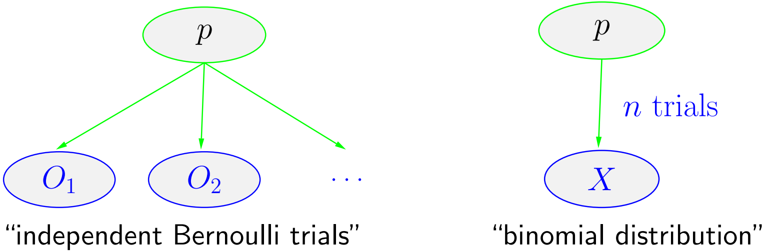
$$\propto f(O_1, O_2, \ldots \,|\, p) \cdot f_0(p)$$

$$P(p_i \,|\, X, n)$$
$$f(p \,|\, X, n)$$

$$\propto f(X \,|\, n, p) \cdot f_0(p)$$

Inferring Bernoulli's trial parameter p

Making several independent trials *assuming* the same p



$P(p_i \,|\, O_1, O_2, \ldots)$ $\quad$ $P(p_i \,|\, X, n)$

$f(p \,|\, O_1, O_2, \ldots)$ $\quad$ $f(p \,|\, X, n)$

Are the two inferences the same?

(not obvious in principle)

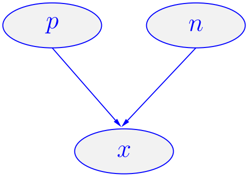
n independent Bernoulli processes

General case

n independent Bernoulli processes

General case

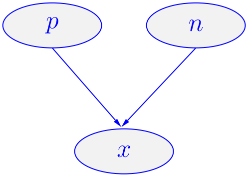
Model



n independent Bernoulli processes

General case

Model



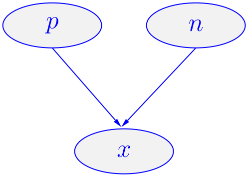
Joint pdf (omitting background condition I):

$$f(x, p, n) = f(x \mid p, n)$$

n independent Bernoulli processes

General case

Model



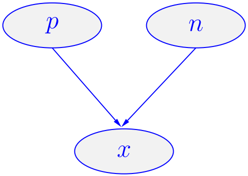
Joint pdf (omitting background condition I):

$$f(x, p, n) \quad = \quad f(x \,|\, p, n) \cdot f(p, n)$$

n independent Bernoulli processes

General case

Model



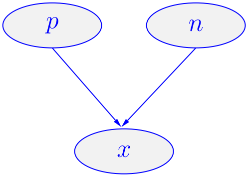
Joint pdf (omitting background condition I):

$$
\begin{aligned}
f(x, p, n) &= f(x \,|\, p, n) \cdot f(p, n) \\
&= f(x \,|\, p, n) \cdot f(p \,|\, n) \cdot f(n)
\end{aligned}
$$

n independent Bernoulli processes

General case

Model



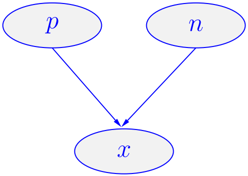
Joint pdf (omitting background condition I):

$$
\begin{aligned}
f(x, p, n) &= f(x \,|\, p, n) \cdot f(p, n) \\
&= f(x \,|\, p, n) \cdot f(p \,|\, n) \cdot f(n) \\
&= f(x \,|\, p, n) \cdot f(n \,|\, p) \cdot f(p)
\end{aligned}
$$

n independent Bernoulli processes

General case

Model



Joint pdf (omitting background condition I):

$$
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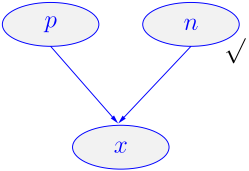
f(x, p, n) &= f(x \,|\, p, n) \cdot f(p, n) \\
&= f(x \,|\, p, n) \cdot f(p \,|\, n) \cdot f(n) \\
&= f(x \,|\, p, n) \cdot f(n \,|\, p) \cdot f(p) \\
&= f(x \,|\, p, n) \cdot f(p) \cdot f(n)
\end{aligned}
$$

(n and p are independent)

n independent Bernoulli processes

Usual case → *n* fixed (for the moment)

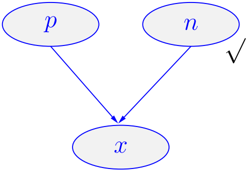
Model



n independent Bernoulli processes

Usual case → n fixed (for the moment)

Model



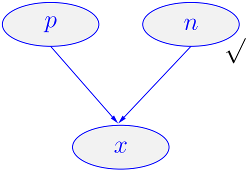
Joint pdf

$$f(x, p \mid n) \;=\; f(x \mid p, n) \cdot f(p)$$

n independent Bernoulli processes

Usual case → n fixed (for the moment)

Model



Joint pdf

$$f(x, p \,|\, n) \;=\; f(x \,|\, p, n) \cdot f(p)$$

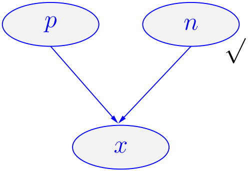
Typical problems

- p is assumed → interested in $f(x \,|\, n, p)$
  → well known binomial;

n independent Bernoulli processes

Usual case → n fixed (for the moment)

Model



Joint pdf

$$f(x, p \mid n) \;=\; f(x \mid p, n) \cdot f(p)$$

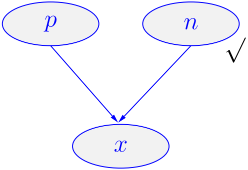
Typical problems

- p is assumed → interested in $f(x \mid n, p)$
  → well known binomial;
- x is assumed ('observed')

n independent Bernoulli processes

Usual case → n fixed (for the moment)

Model



Joint pdf

$$f(x, p \mid n) \;=\; f(x \mid p, n) \cdot f(p)$$

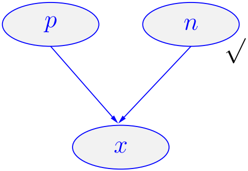
Typical problems

- p is assumed → interested in $f(x \mid n, p)$
  → well known binomial;
- x is assumed ('observed') → $f(p \mid n, x)$

n independent Bernoulli processes

Usual case → n fixed (for the moment)

Model



Joint pdf

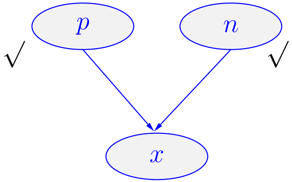
$$f(x, p \mid n) \;=\; f(x \mid p, n) \cdot f(p)$$

Typical problems

- p is assumed → interested in $f(x \mid n, p)$
  → well known binomial;
- x is assumed ('observed') → $f(p \mid n, x)$:
  → ?

n independent Bernoulli processes

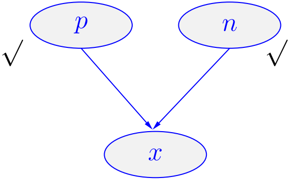
Graphical models of the typical problems



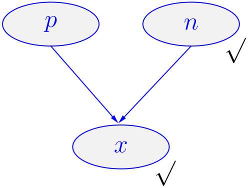
$\rightarrow\; f(x \mid n, p)$

n independent Bernoulli processes

Graphical models of the typical problems



$\rightarrow f(x \,|\, n, p)$



$\rightarrow f(p \,|\, n, x)$

The End