Measurements, uncertainties and probabilistic inference/forecasting

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Università di Roma La Sapienza e INFN
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‘**Probabilistic**’ stressed

From the previous lectures it should be clear that I stick to

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At every step 'probability' means...

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At every step ‘probability’ means... **probability**.

Although other ‘methods’ might be more popular

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[Plus other **prescriptions** you might imagine. . .]

n independent Bernoulli processes

General case

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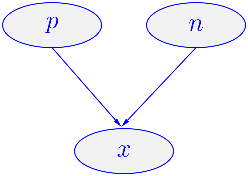
General case

Model



n independent Bernoulli processes

General case

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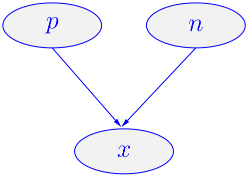

Joint pdf (omitting background condition I):

$$f(x, p, n) \;=\; f(x \mid p, n)$$

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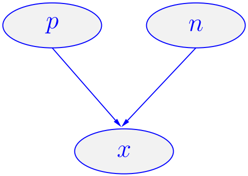

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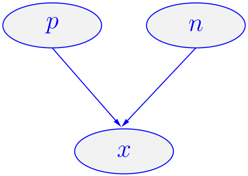

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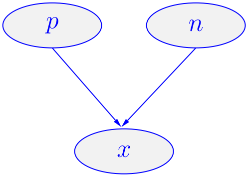

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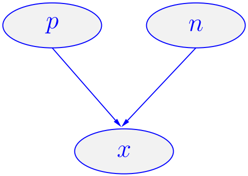

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(n and p are independent)

n independent Bernoulli processes

Usual case → n fixed (for the moment)

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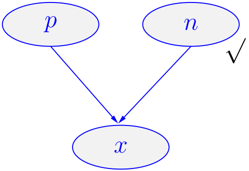

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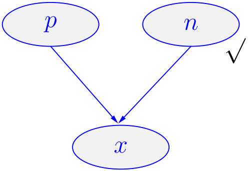

Joint pdf

$$f(x, p \mid n) \;=\; f(x \mid p, n) \cdot f(p)$$

n independent Bernoulli processes

Usual case $\rightarrow$ n fixed (for the moment)

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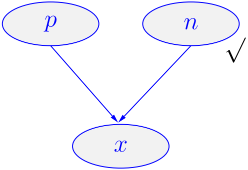

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Typical problems

- p is assumed $\rightarrow$ interested in $f(x \mid n, p)$
  $\rightarrow$ well known binomial;

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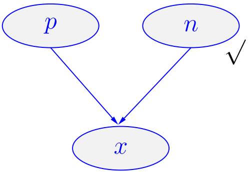

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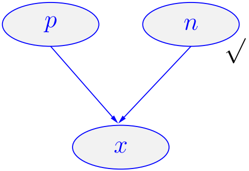

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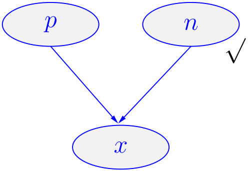

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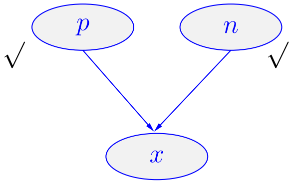
  → ?

n independent Bernoulli processes

Graphical models of the typical problems



$$\rightarrow\ f(x \,|\, n, p)$$

n independent Bernoulli processes

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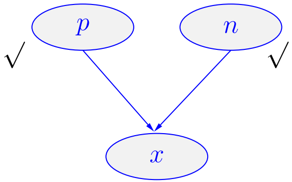

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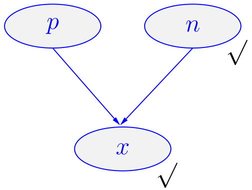

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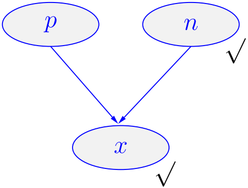
n independent Bernoulli processes

Inferring p



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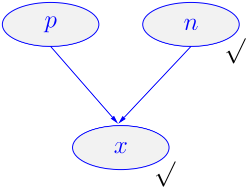

$$f(p\,|\,x,n) \;=\; \frac{f(p,x\,|\,n)}{f(x\,|\,n)}$$

n independent Bernoulli processes

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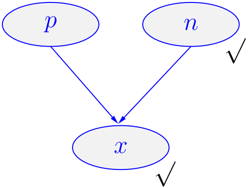

$$f(p\,|\,x,n) = \frac{f(p,x\,|\,n)}{f(x\,|\,n)} = \frac{f(x\,|\,n,p)\cdot f_0(p)}{f(x\,|\,n)}$$

n independent Bernoulli processes

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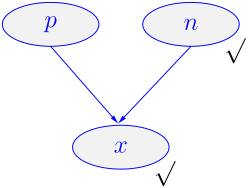

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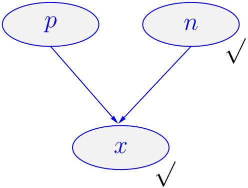

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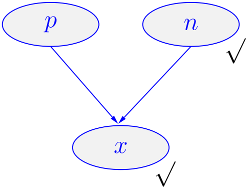

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(denominator just normalization!)

Inferring "Bernoulli's p"

We just need to make explicit $f(x \,|\, n, p)$:

$$f(x \,|\, n, p) \;=\; \binom{n}{x}\, p^x\, (1-p)^{n-x}$$

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We get then, including normalization:

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(The binomial coefficient is irrelevant, not depending on p)

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For teaching purposes we start from a **uniform prior**, i.e. $\boldsymbol{f_\circ(p) = 1}$:

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- The integral at the denominator is the special function "β" (also defined for real values of x and n).
- In our case these two numbers are integer and the integral becomes equal to

$$\frac{x!\,(n-x)!}{(n+1)!}$$

Inferring "Bernoulli's p"

Solution for uniform prior (think to **Bayes' billard**)

$$f(p\,|\,x,n) \;=\; \frac{(n+1)!}{x!\,(n-x)!}\,p^x\,(1-p)^{n-x}$$

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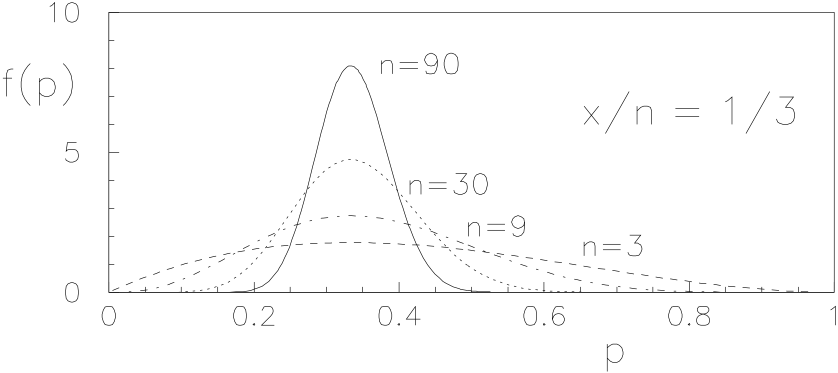

Inferring "Bernoulli's p"

Summaries of the **posterior distribution**

$$p_m = \text{mode}(p) \quad = \quad \frac{x}{n}$$

Inferring "Bernoulli's p"

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"recursive Laplace formula"

("Laplace's rule of succession")

Inferring "Bernoulli's p"

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"recursive Laplace formula"

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$$\begin{aligned}\text{Var}(p) &= \frac{(x+1)(n-x+1)}{(n+3)(n+2)^2} \\ &= \frac{x+1}{n+2}\left(\frac{n+2}{n+2} - \frac{x+1}{n+2}\right)\frac{1}{n+3} \\ &= \text{E}(p)\,(1-\text{E}(p))\,\frac{1}{n+3}\end{aligned}$$

Inferring the "Bernoulli's p"

About the meaning of E(p)

- We have used the "first"$^{(*)}$ n trials to learn about "p".

  [$^{(*)}$ "First" does not imply time order, but just order in usage.]

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$$P(E_{i>n}) = ??$$

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Inferring the “Bernoulli's p”

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Inferring the "Bernoulli's p"

About the meaning of $\mathrm{E}(p)$

- We have used the "first"$^{(*)}$ n trials to learn about "p".
  [$^{(*)}$ "First" does not imply time order, but just order in usage.]
- What will be the probability of other trials?

$$P(E_{i>n}) = ??$$

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(But keep in mind the **inductivist turkey**!)

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Large number behaviour

When the number of successes and the number of failures become 'large' (x large is not enough, as it can be easily understood from the simmetric properties of the binomial $p \leftrightarrow q$):

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(Similarly to Bernoulli's theorem, it is not a 'mathematical' limit!)

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— **No need of 'frequentistic definition'!**

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Frequency and probability are **related** in probability theory:

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BUT

- There is no need to identify the two concepts.
- It does not justify the frequentistic definition.

Propagation of **errors** in the evaluation of efficiency

From a recent 'tesi di laurea' in Rome ('quadriennale') (undergraduate thesis)

Da questa analisi si ottengono

$$\mathrm{N} = (82502 \pm 287) \qquad \mathrm{n_S} = (82378 \pm 287).$$

dove $\sigma_{N(n)} = \sqrt{N(n)}$.

Da N e $\mathrm{n_S}$ si ricava il valore dell'efficienza in Pos 1:

$$\epsilon_{\mathrm{S(Pos1)}} = \frac{\mathrm{n_S}}{\mathrm{N}} = (99.847)\%$$

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$$\frac{\sigma(\epsilon)^{wrong}}{\sigma(\epsilon)^{correct}} = \frac{1/\sqrt{N}\sqrt{n/N\cdot(1+n/N)}}{1/\sqrt{N}\sqrt{n/N\cdot(1-n/N)}}$$

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$$= 36$$

Propagation of errors. . . and of mistakes

Eseguendo queste operazioni otteniamo il seguente risultato:

$$\epsilon_{S(\mathrm{Pos1})} = (99.847 \pm 0,005^{(\mathrm{stat})} \pm 0,010^{(\mathrm{sist})})\%.$$

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Good luck to the experiment!

Inferring the "Bernoulli's p"

Approximate solution using the 'Gaussian trick'

Exercise

- Given $f(p) \propto p^x\,(1-p)^{n-x}$,
- define $\varphi(p) = -\ln f(p)$
- and evaluate
  - $\frac{d\varphi}{dp}$
  - $\frac{d^2\varphi}{dp^2}$
- Then estimate
  - $\mathrm{E}(p) \approx p_m$ from minimum;
  - $\sigma^2(p)$ from second derivative at the minimum.

Inferring "Bernoulli's p"

Observing $x = \mathbf{0}$

$$f(0\,|\,\mathcal{B}_{n,p}) \;=\; (1-p)^n,$$

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$$\begin{aligned} f(0\,|\,\mathcal{B}_{n,p}) &= (1-p)^n, \\ f(p\,|\,x=0,n,\mathcal{B}) &= \frac{(1-p)^n}{\int_0^1 (1-p)^n\,dp} = (n+1)\,(1-p)^n, \end{aligned}$$

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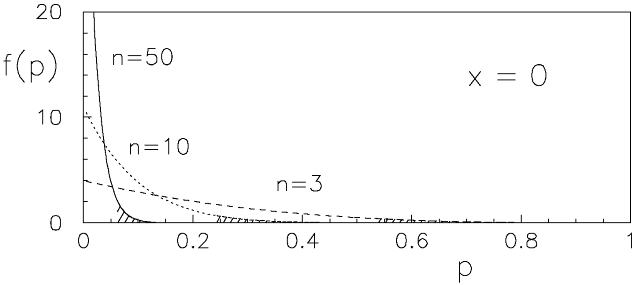
f(p\,|\,x=0,n,\mathcal{B}) &= \frac{(1-p)^n}{\int_0^1 (1-p)^n\,dp} = (n+1)\,(1-p)^n, \\
F(p\,|\,x=0,n,\mathcal{B}) &= 1-(1-p)^{n+1}\,.
\end{aligned}
$$



Inferring "Bernoulli's p"

Observing $x = \mathbf{0}$

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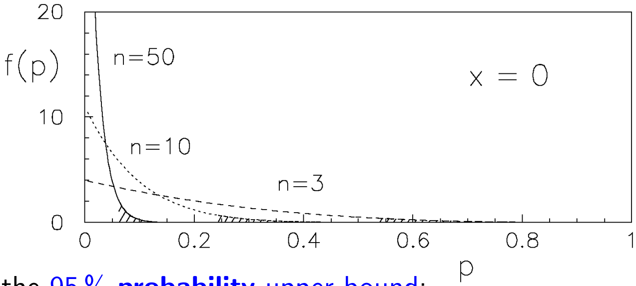

To get the 95 % **probability** upper bound:

$$F(p_\circ\,|\,x=0,n,\mathcal{B}) = 0.95,$$

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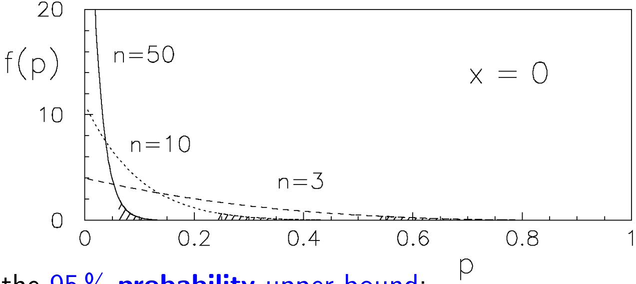

To get the 95 % **probability** upper bound:

$$
\begin{aligned}
F(p_\circ\,|\,x=0,n,\mathcal{B}) &= 0.95, \\
p_\circ &= 1-\sqrt[n+1]{0.05}\,.
\end{aligned}
$$

Inferring "Bernoulli's p"

Observing $x = n$

$$f(n \,|\, \mathcal{B}_{n,p}) = p^n$$

Inferring “Bernoulli's p”

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Inferring the "Bernoulli's p"

A glance to upper/lower probabilistic limits

	Probability level $= 95\,\%$		
n	$x = n$	$x = 0$	
	binomial	binomial	Poisson approx. ($p_\circ = 3/n$)
3	$p \geq 0.47$	$p \leq 0.53$	$p \leq 1$
5	$p \geq 0.61$	$p \leq 0.39$	$p \leq 0.6$
10	$p \geq 0.76$	$p \leq 0.24$	$p \leq 0.3$
50	$p \geq 0.94$	$p \leq 0.057$	$p \leq 0.06$
100	$p \geq 0.97$	$p \leq 0.029$	$p \leq 0.03$
1000	$p \geq 0.997$	$p \leq 0.003$	$p \leq 0.003$

Upper/lower probability bounds – caveat!

The probabilistic upper/lower bounds of the previous slides depend on the assumption $f(p) = 1$

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- ... but **it isn't!**
- Imagine that p refers to a branching ratio:
  $f_0(p) = 1$ implies

$$P(p \leq 0.1) = P(p \geq 0.9)$$

$$P(p \leq 0.01)\,[= P(p \geq 0.99)]$$

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\cdots &\quad \cdots
\end{aligned}
$$

Really do you believe so?

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Really do you believe so?

Exercise: try to plot $f(p \,|\, x = 0, n = 100)$ in log-log scale

```
> p=10^seq(-5,-1,len=100);
> plot(p, (1-p)^100, ty='l', log='xy'); grid()
```

(and think about it!)

Very rare processes

Sensitivity bounds: some hints for **self study**

Let us restart from the Bayes' rule

$$f(p \,|\, x, n) \;\;\propto\;\; p^x\,(1-p)^{n-x}\,f_\circ(p)$$

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- Do the math and calculate the posterior.
- Anticipation of the result
  - if the prior is not updated at all, or if it is not changed significantly, than the experimental information is irrelevant.

Inferring the "Bernoulli's p"

Mathematically convenient priors

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Beta distribution

Indeed, such a pdf exists ($a = r - 1$; $b = s - 1$).

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In general, given the generic uncertain number X,

$$f(x\,|\,\mathrm{Beta}(r,s)) = \frac{1}{\beta(r,s)}x^{r-1}(1-x)^{s-1} \qquad \begin{cases} r,\, s > 0 \\ 0 \le x \le 1 \end{cases}$$

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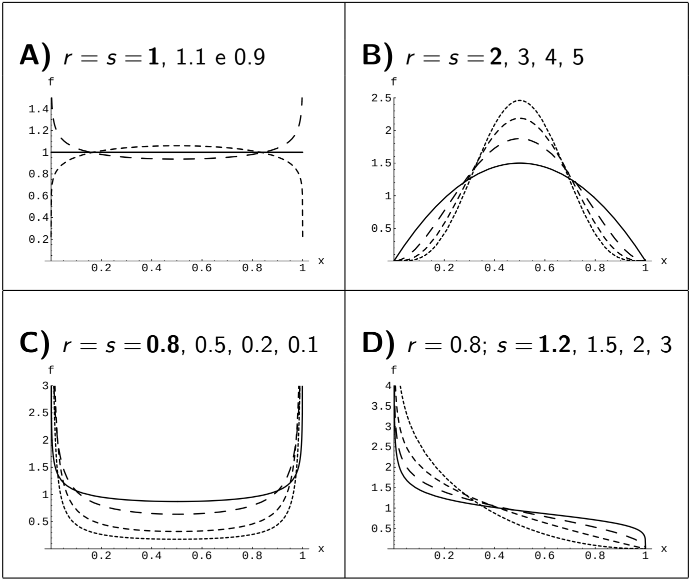
Try e.g.

```
> p<-seq(0,1,by=0.01)
> plot(p, dbeta(p, 3, 5), ty='l', col='blue')
```

Beta distribution

Some examples



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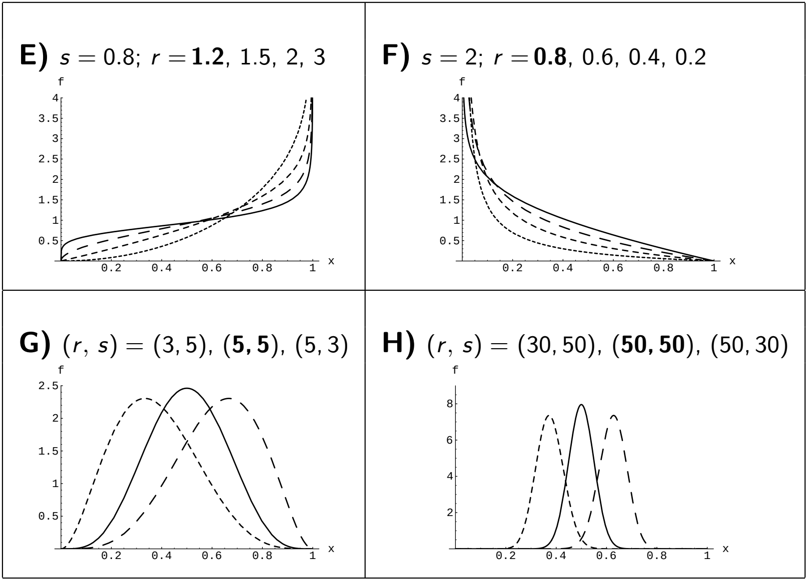


Beta distribution

Summaries

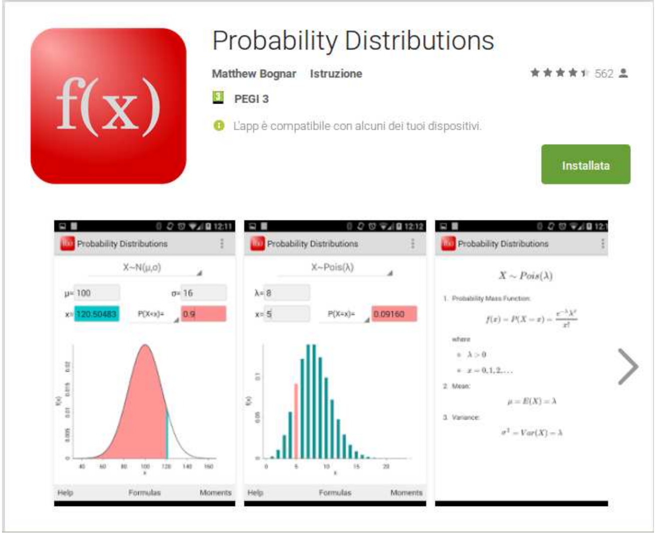
$$\begin{aligned} \mathrm{E}(X) &= \frac{r}{r+s} \\ \mathrm{Var}(X) &= \frac{rs}{(r+s+1)\,(r+s)^2}\,. \end{aligned}$$

Mode, unique if $r > 1$ and $s > 1$:

$$\frac{r-1}{r+s-2}$$

A useful app

https://play.google.com/store/apps/details?id=com.mbognar.probdist



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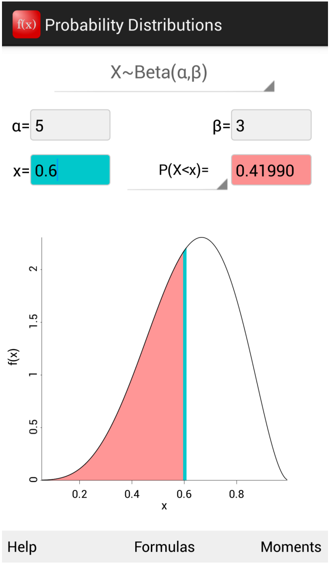

Beta distribution as prior

Let us finally apply it to infer the Bernoulli's p

$$\begin{aligned} f(p \mid n, x, \text{Beta}(r_i, s_i)) &\propto \left[p^x(1-p)^{n-x}\right] \times \left[p^{r_i-1}(1-p)^{s_i-1}\right] \\ &\propto p^{x+r_i-1}(1-p)^{n-x+s_i-1}\,. \end{aligned}$$

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Simple updating rule:

$$
\begin{aligned}
r_f &= r_i + x \\
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$$\mathrm{E}(X) = \frac{r_f}{r_f + s_f} = \frac{x+1}{n+2}$$

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$$\mathrm{mode}(X) = \frac{r_f - 1}{r_f + s_f - 2} = \frac{x}{n}$$

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- not all *conjugate priors* are as flexible as the Beta.

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- its parameters are updated by the the 'likelihood'.

Note:

- not all *conjugate priors* are as flexible as the Beta.

  (In particular, the Gaussian is self-conjugate,
  which is not so great...)

More on priors

Data dominated inference

Let's look again at how the prior gets updated

$$\begin{aligned} f(p \mid n, x, r_i, s_i) &\propto \left[p^x (1-p)^{n-x}\right] \times \left[p^{r_i-1}(1-p)^{s_i-1}\right] \\ &\propto p^{x+r_i-1}(1-p)^{n-x+s_i-1} \end{aligned}$$

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\mathrm{E}(p) &= \frac{r_f}{r_f+s_f} = \frac{r_i+x}{r_i+s_i+n} \\
\mathrm{Var}(p) &= \frac{r_f\,s_f}{(r_f+s_f+1)\,(r_f+s_f)^2} \\
&= \frac{(r_i+x)\cdot(s_i+n-x)}{(r_i+s_i+n+1)\,(r_i+s_i+n)^2}
\end{aligned}
$$

If $x \gg r_i$ and $(n-x) \gg s_i$

$$
\begin{aligned}
r_f &\approx x \\
s_f &\approx (n-x)
\end{aligned}
$$

Predictive distribution

Predicting future nr. of successes and future frequences

- Imagine we have have got 5 successes in 10 trials.

Predictive distribution

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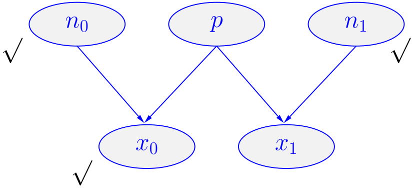
- But we are not sure about it: we need to take into account all possible values, each weighted by $f(p)$

Predictive distribution

Predicting future nr. of successes and future frequences



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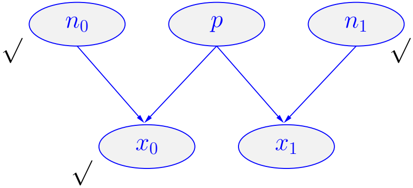

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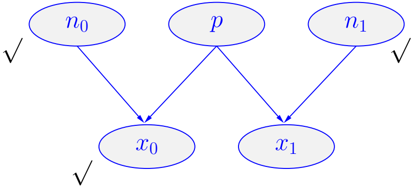

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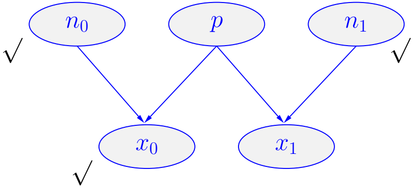

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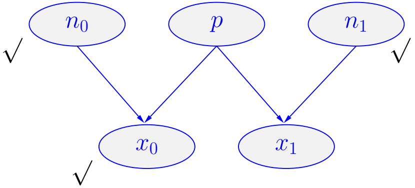
$$f(x_1 \mid n_1, n_0, x_0) = \int_0^1 f(x_1 \mid n_1, p)\, f(p \mid x_0, n_0)\, dp$$

- $X_1 \to f_1$

Predictive distribution

Predicting future nr. of successes and future frequences



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- $X_1 \rightarrow f_1$ (Predicting a **future frequency from a past frequency**)

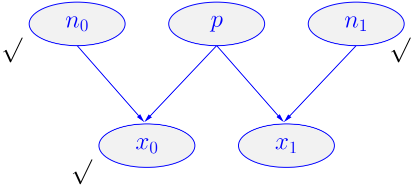
Predictive distribution

Some examples

$f(x_1 \mid n_0, x_0, n_1 = 10)$ in %

X_1	$\frac{X_1}{n_1}$	$\begin{cases} x_0 = 1 \\ n_0 = 2 \end{cases}$	$\begin{cases} x_0 = 10 \\ n_0 = 20 \end{cases}$	$\begin{cases} x_0 = 100 \\ n_0 = 200 \end{cases}$	$\begin{cases} x_0 = 1000 \\ n_0 = 2000 \end{cases}$
0	0	3.85	0.42	0.12	0.10
1	0.1	6.99	2.29	1.11	0.99
2	0.2	9.44	6.51	4.67	4.42
3	0.3	11.19	12.54	11.88	11.74
4	0.4	12.24	18.07	20.21	20.48
5	0.5	12.59	20.33	24.02	24.55
6	0.6	12.24	18.07	20.21	20.48
7	0.7	11.19	12.54	11.88	11.74
8	0.8	9.44	6.51	4.67	4.42
9	0.9	6.99	2.29	1.11	0.99
10	1	3.84	0.42	0.12	0.10
$\mathrm{E}(X_1)$		5	5	5	5
$\sigma[X_1]$		2.64	1.87	1.62	1.58

Joint inference and prediction



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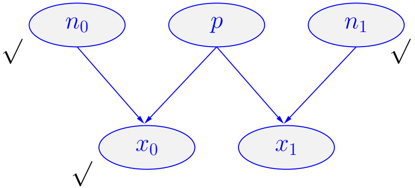

In reality the general solution starts from

$$f(n_0, p, n_1, x_0, x_1)$$

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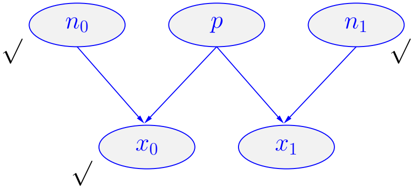

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conditioning on what it is 'known' (or 'assumed'):

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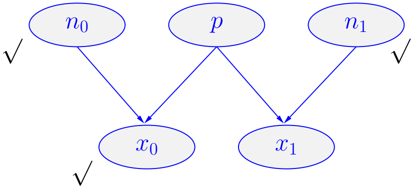

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$\Rightarrow$ p **and** x_1 **are correlated!**

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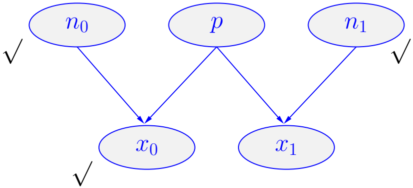

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$$\rho(p, x_1) > 0$$

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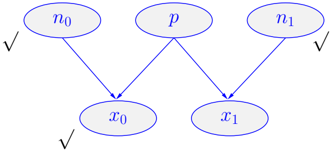
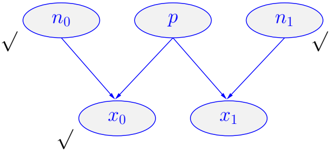


Let's do the math.

Joint inference and prediction



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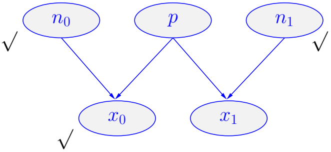
- Three **observed variables**

Joint inference and prediction



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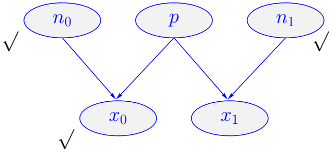
- Three **observed variables** (no uncertainty): n_0, x_0 and n_1.

Joint inference and prediction



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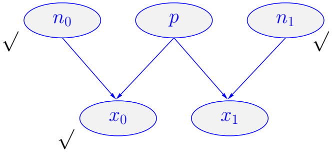

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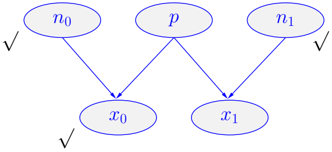

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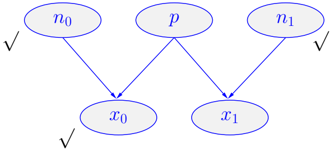

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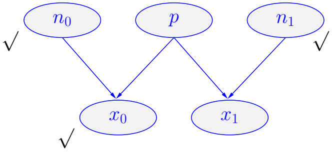

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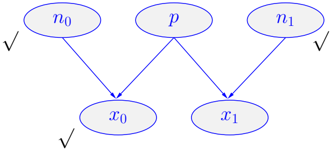

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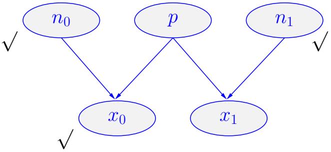
$$\propto f(p, x1, n_0, n_1, x_0)$$

$$\tilde{f}(p, x_1 \,|\, n_0, x_0, n_1) = f(p, x1, n_0, n_1, x_0)$$

$\tilde{f}()$: unnormalized pdf.

Joint inference and prediction



Using the chain rule ('bottom-up')
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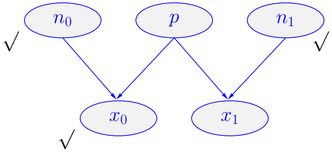

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$$
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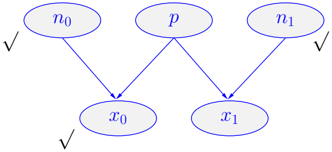

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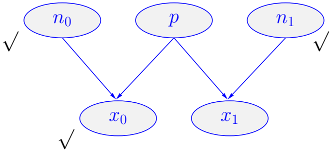

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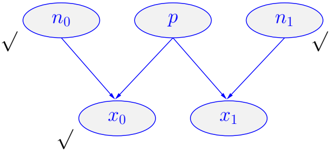
Problem almost solved

Joint inference and prediction



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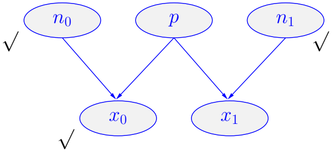
- Possibly calculate the normalization, then all moments and probability intervals of interest.

Joint inference and prediction



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- Do it numerically,

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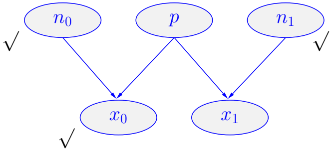

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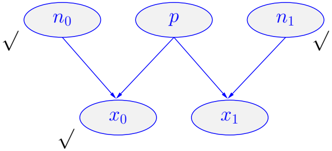
Problem almost solved

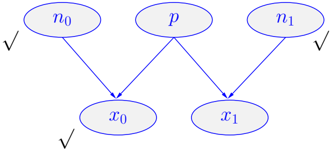
- Possibly calculate the normalization, then all moments and probability intervals of interest.
- Do it numerically,
- or by by sampling.

Joint inference and prediction



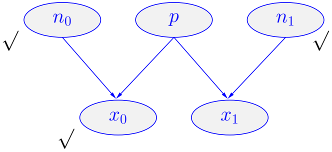
$\Rightarrow$ sample $\tilde{f}(p, x_1 \,|\, n_0, x_0, n_1)$

Joint inference and prediction



$\Rightarrow$ sample $\tilde{f}(p, x_1 \,|\, n_0, x_0, n_1)$ using Monte Carlo techniques

Joint inference and prediction



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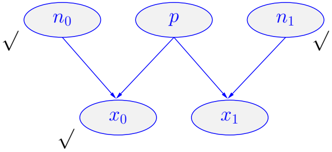
Joint inference and prediction



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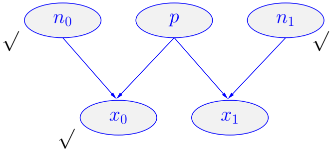
Joint inference and prediction



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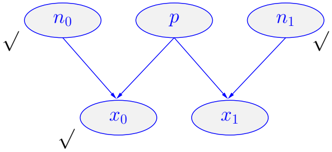
- by Gibbs sampler (JAGS: Just Another Gibbs Sampler) if pdf's involved allow it;

Joint inference and prediction



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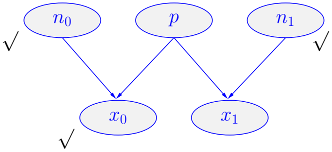
- by Metropolis

Joint inference and prediction



⇒ sample $\tilde{f}(p, x_1 \,|\, n_0, x_0, n_1)$ using Monte Carlo techniques

⇒ Markov Chain Monte Carlo (MCMC)

⇒ JAGS does it for us

- by Gibbs sampler (JAGS: Just Another Gibbs Sampler) if pdf's involved allow it;
- by Metropolis ("when the going gets tough, the tough get going" – J. Belushi)

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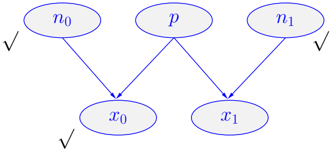

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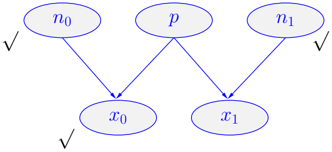

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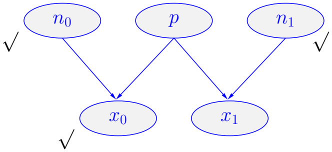
(No details on MCMC provided $\longrightarrow$ see references on the web site)

Graphical models: some terminology



- nodes (observed/unobserved);
- child/childred;
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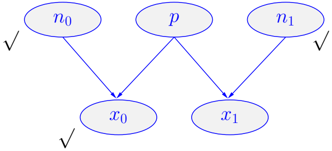
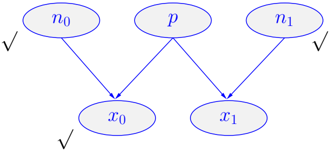


- nodes (observed/unobserved);
- child/childred;
- parent(s).
- **A node without parents needs a prior**
  (node *p* in this case)

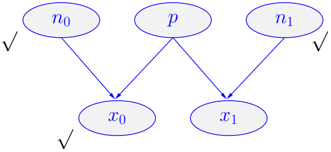
Joint inference and prediction in JAGS



Model

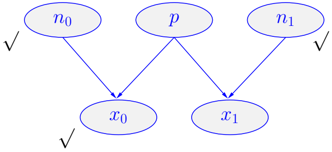
```
model{
    x0 ~ dbin(p, n0);
    x1 ~ dbin(p, n1);
     p ~ dbeta(1, 1);
}
```

Joint inference and prediction in JAGS



Then the model has to be in a file.

Joint inference and prediction in JAGS



Then the model has to be in a file.
For such a small model we can write it directly from R on a temporary file:

```
model = "tmp_model.bug"
write("
model{
    x0 ~ dbin(p, n0);
    x1 ~ dbin(p, n1);
     p ~ dbeta(1, 1);
}
", model)
```

Use of JAGS from R via rjags

Second part of the R script (⇒ inf_p_pred.R)

```
library(rjags)

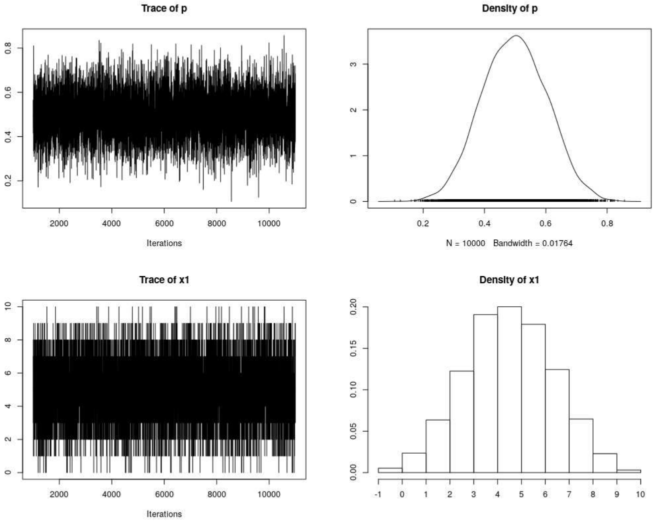
data = list(n0=20, x0=10, n1=10)

jm <- jags.model(model, data)
chain <- coda.samples(jm, c("p", "x1"), n.iter=10000)

plot(chain)
print(summary(chain))
```

Use of JAGS from R via rjags

$(n0 = 20,\ x0 = 10,\ n1 = 10)$



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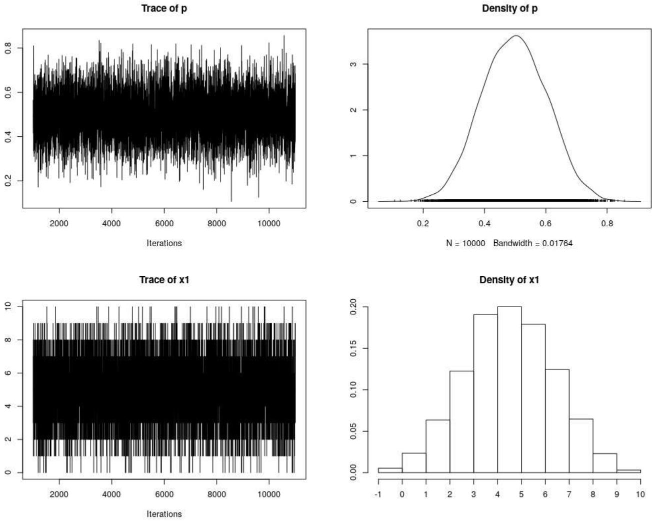

$p = 0.498 \pm 0.105$; $x_1 = 4.98 \pm 1.86$ (10000 samples).

Inference and prediction with JAGS/rjags

Comparison with exact result of $f(x_1 \mid n_0, x_0, n_1)$

$f(x_1 \mid n_0, x_0, n_1 = 10)$ in %

X_1	$\frac{X_1}{n_1}$	$\begin{cases} x_0 = 1 \\ n_0 = 2 \end{cases}$	$\begin{cases} x_0 = 10 \\ n_0 = 20 \end{cases}$	$\begin{cases} x_0 = 100 \\ n_0 = 200 \end{cases}$	$\begin{cases} x_0 = 1000 \\ n_0 = 2000 \end{cases}$
0	0	3.85	0.42	0.12	0.10
1	0.1	6.99	2.29	1.11	0.99
2	0.2	9.44	6.51	4.67	4.42
3	0.3	11.19	12.54	11.88	11.74
4	0.4	12.24	18.07	20.21	20.48
5	0.5	12.59	20.33	24.02	24.55
6	0.6	12.24	18.07	20.21	20.48
7	0.7	11.19	12.54	11.88	11.74
8	0.8	9.44	6.51	4.67	4.42
9	0.9	6.99	2.29	1.11	0.99
10	1	3.84	0.42	0.12	0.10
$E(X_1)$		5	5	5	5
$\sigma[X_1]$		2.64	1.87	1.62	1.58

Inference and prediction with JAGS/rjags

Scatter plot of sampled $f(p, x_1 \mid n_0, x_0, n_1)$

```
p  <- as.vector(chain[[1]][,1])
x1 <- as.vector(chain[[1]][,2])
plot(x1, p, col='blue',
     main=sprintf("cor(p,x1) = %.2f", cor(p,x1)))
print( table(x1)/10000 )
```

Inference and prediction with JAGS/rjags

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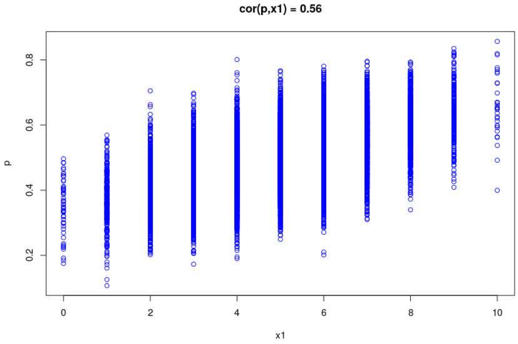

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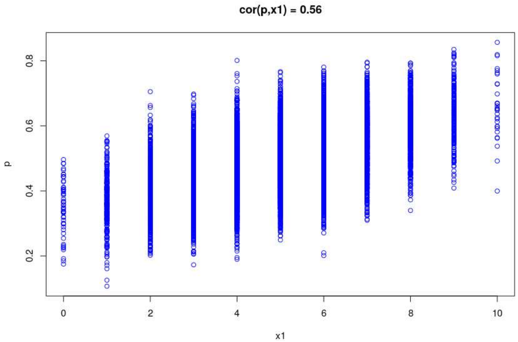
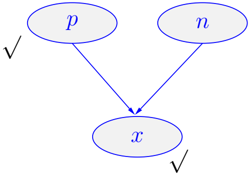


(The last command, print(. . .), produces the relative frequencies of occurrance of x_1 → **try it**)

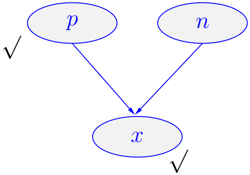
n independent Bernoulli processes

Inferring n



n independent Bernoulli processes

Inferring *n*



Think at a detector having a *well known efficiency* ($\epsilon \equiv p$)

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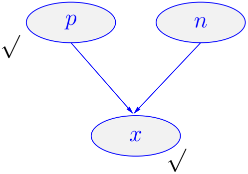

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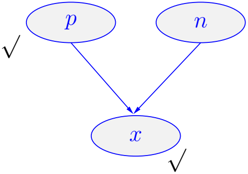

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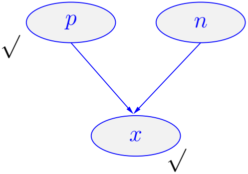

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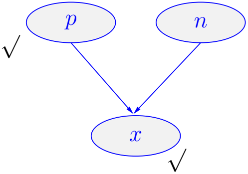

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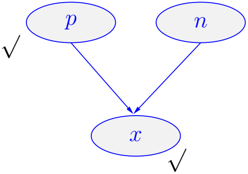

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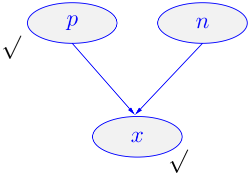

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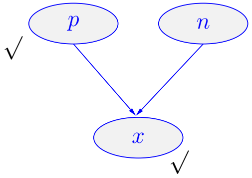
  [or, more precisely, what is the rate r? $\longrightarrow f(r \mid x, T)$ **?**]

n independent Bernoulli processes

Extending the model

Our problem (but in Physics it is often not so simple)



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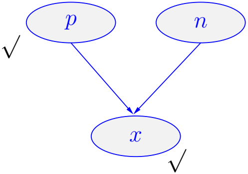

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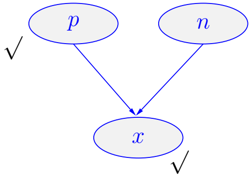

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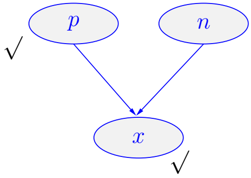

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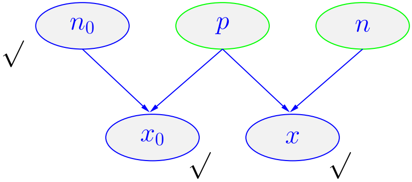

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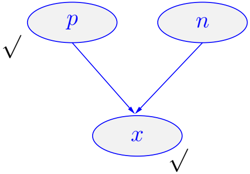

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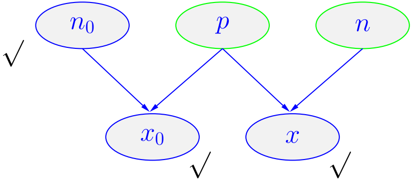

But what is n?

Extending the model

In Physics we are usually not interested in the numbers we do see, but in those which have 'physical meaning'.

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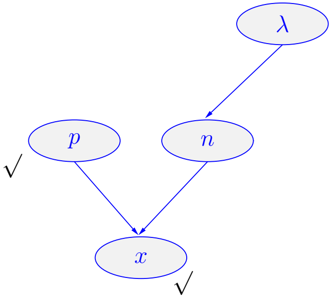

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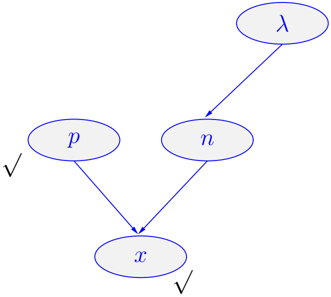

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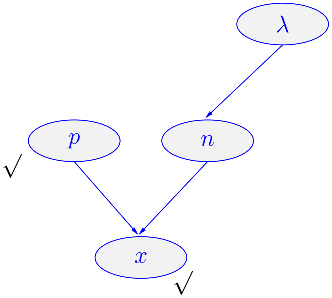

But, as we have seen studying the **Poisson process**,
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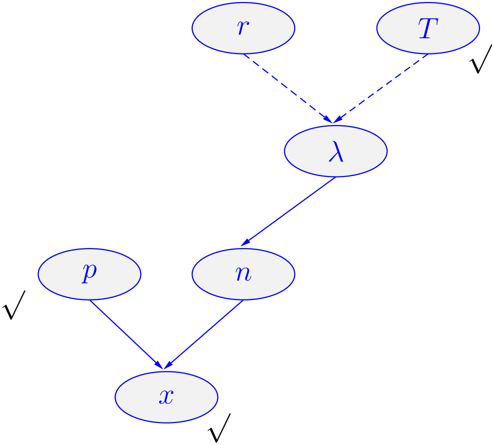
Extending the model

$\boldsymbol{\lambda} = r \cdot T$:



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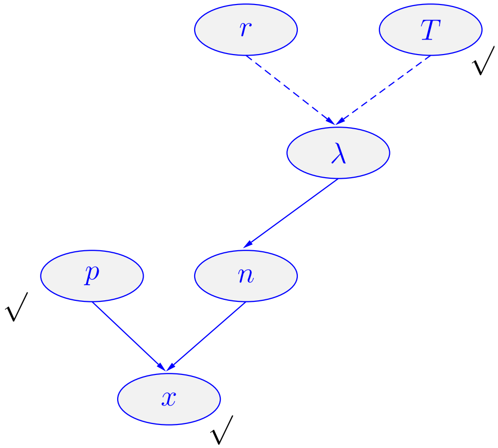

(Dashed arrows used in literature for deterministic links)

Extending the model

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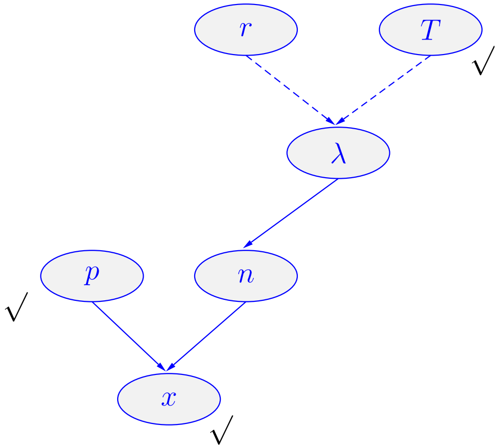
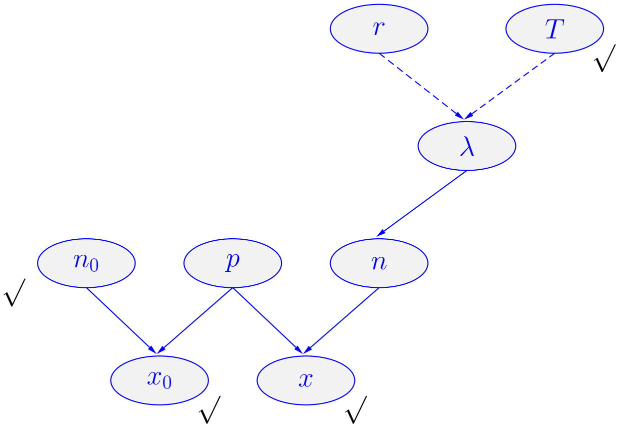


(Dashed arrows used in literature for deterministic links)

In JAGS, e.g., `lambda <- r * T;`

Extending the model

Remembering that p was got from a measurement:

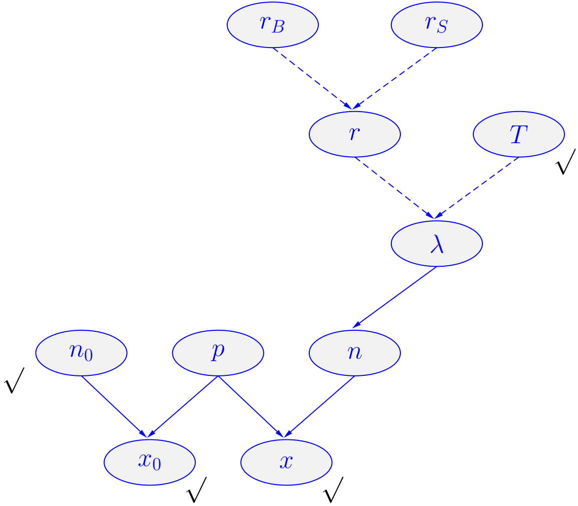


Extending the model

The rate r gets contributions from signal and background

Extending the model

The rate *r* gets contributions from signal and background



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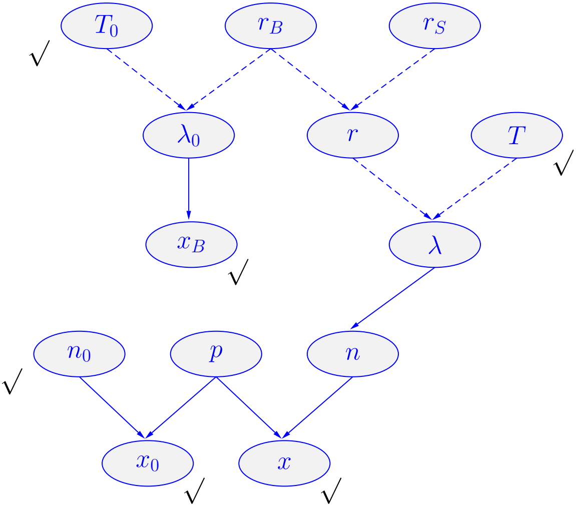

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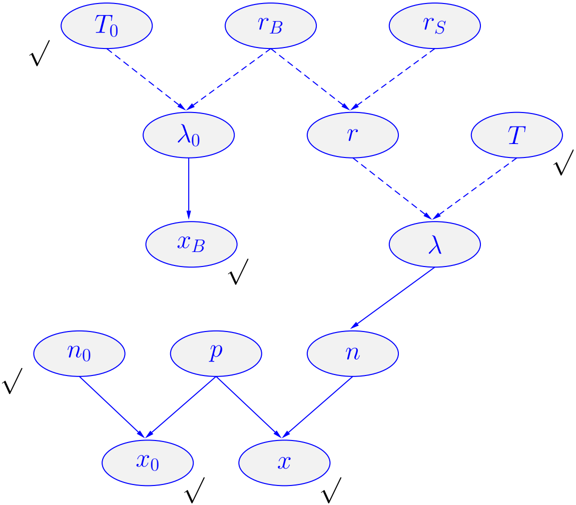
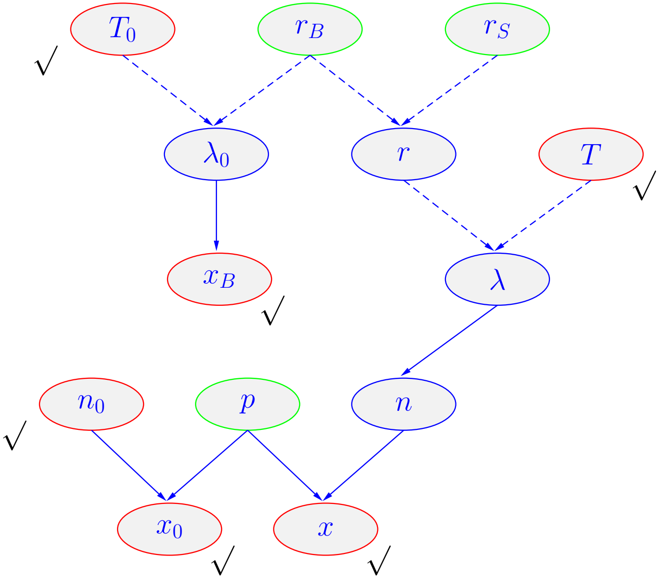


(T_0 and T assumed to be measured with sufficient accuracy)

Extending the model



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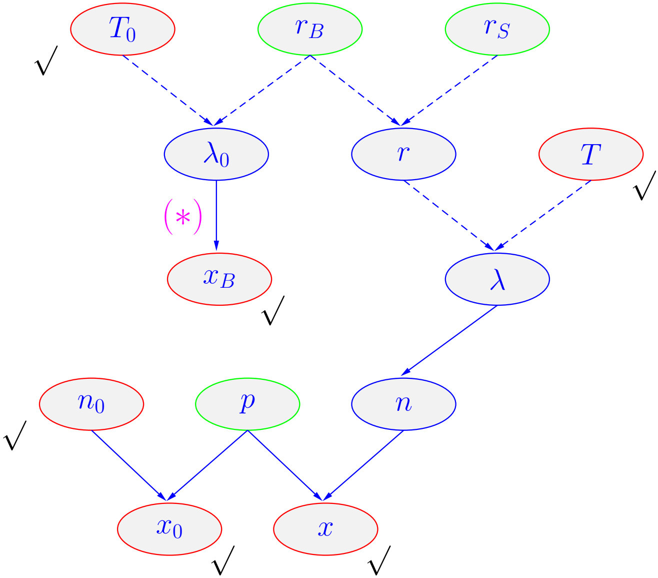
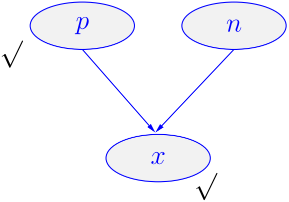


(*) Assuming unity efficiency

Inferring n 'assuming' p and x

Back to our initial problem



Inferring n 'assuming' p and x

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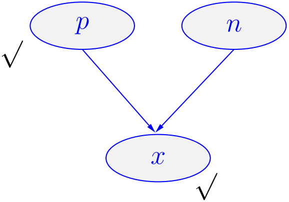

$$f(n \,|\, p, x) \;\; \propto \;\; f(x \,|\, n, p) \cdot f_0(n)$$

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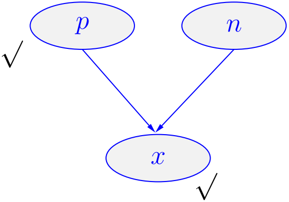

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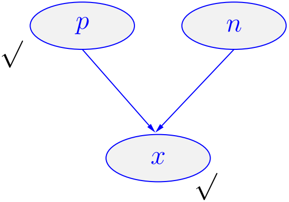

$$
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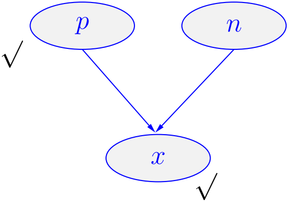
&\propto f(x \,|\, n, p) \qquad [\,\text{uniform prior}\,] \\
&\propto \frac{n!}{x!\,(n-x)!}\, p^x \cdot (1-p)^{n-x}
\end{aligned}
$$

Inferring n 'assuming' p and x

Back to our initial problem



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\begin{aligned}
f(n\,|\,p,x) &\propto f(x\,|\,n,p)\cdot f_0(n) \\
&\propto f(x\,|\,n,p) \qquad\qquad [\,\text{uniform prior}\,] \\
&\propto \frac{n!}{x!\,(n-x)!}\,p^x\cdot(1-p)^{n-x} \\
&\propto \frac{n!}{x!\,(n-x)!}\,p^x\cdot\frac{(1-p)^n}{(1-p)^x}
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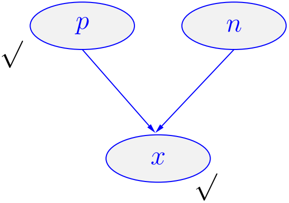

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&\propto \frac{n!}{(n-x)!}\,(1-p)^n
\end{aligned}
$$

Inferring n 'assuming' p and x

Example in R with $p = 0.75$ and $x = 10$

```
p = 0.75; x = 10
n.max = 30

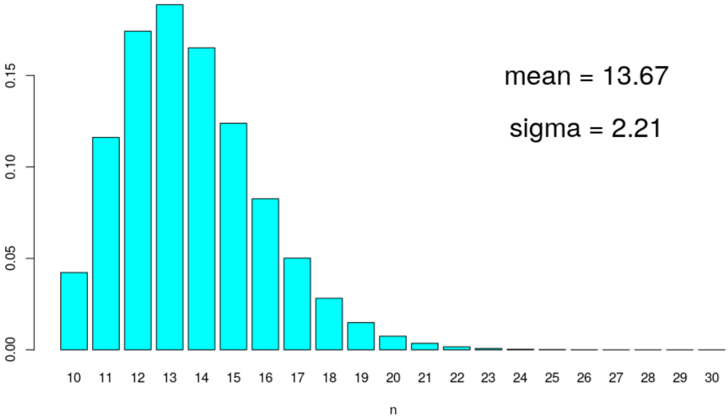
n  = x:n.max
fn = factorial(n)/factorial(n-x)*(1-p)^n
fn = fn/sum(fn)

media.n  = sum(fn*n)
media.n2 = sum(fn*n^2)
sigma.n  = sqrt(media.n2 - media.n^2)

barplot(fn, names=n, col='cyan', xlab='n')
text(20,0.15, sprintf("mean = %.2f", media.n), cex=2)
text(20,0.12, sprintf("sigma = %.2f", sigma.n),cex=2)
```

Inferring n 'assuming' p and x

$$f(n \,|\, x = 10, p = 0.75)$$



Inferring n 'assuming' p and x

Or we can feed JAGS with the following simple model

```
model{
    x ~ dbin(p, n);
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- `I(nmin,)` means that `n` cannot be smaller than `nmin`

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- `I(nmin,)` means that `n` cannot be smaller than `nmin` (`nmin` is indeed equal to the observed `x`, for obvious reasons, but JAGS needs a separate constant)
  The remaining R code is left as **exercise**

Inferring Poisson's λ

- set up the problem;

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Inferring Poisson's λ

- set up the problem;
- solution for uniform prior;
- the case of no events observed;
- prior conjugate;
- predictive distribution;
- from λ to r (not covered, since it is straightforward; but remember that the 'physical quantity' is r)

Inferring Poisson's λ

$$f(\lambda \,|\, x, \mathcal{P}) \;=\; \frac{\frac{\lambda^x\, e^{-\lambda}}{x!}\, f_\circ(\lambda)}{\int_0^\infty \frac{\lambda^x\, e^{-\lambda}}{x!}\, f_\circ(\lambda)\,\mathrm{d}\lambda}\,.$$

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Assuming $f_\circ(\lambda)$ constant up to a certain $\lambda_{max} \gg x$ and making the integral by parts we obtain

$$f(\lambda \,|\, x, \mathcal{P}) \;=\; \frac{\lambda^x\, e^{-\lambda}}{x!}$$

$$F(\lambda \,|\, x, \mathcal{P}) \;=\; 1 - e^{-\lambda}\left(\sum_{n=0}^{x} \frac{\lambda^n}{n!}\right)$$

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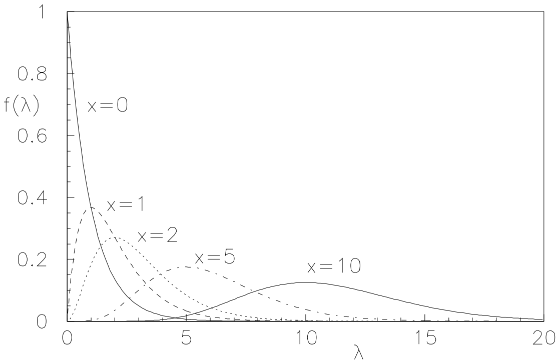
Summaries

$$\mathrm{E}(\lambda) \;=\; x+1,$$
$$\mathrm{Var}(\lambda) \;=\; x+1,$$
$$\lambda_m \;=\; x$$

Some examples of $f(\lambda)$

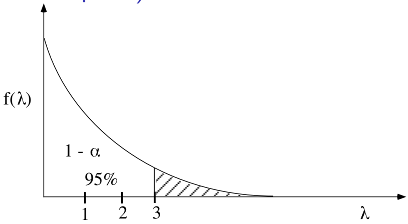


For 'large' x $f(\lambda)$ it becomes Gaussian with expected value x and standard deviation $\sqrt{x}$.

The difference between the most probable λ and its expected value for small x is due to the asymmetry of $f(\lambda)$.

Inferring λ from $x = 0$

(From a flat prior!)



$$
\begin{aligned}
f(\lambda \,|\, x = 0, \mathcal{P}) &= e^{-\lambda} \\
F(\lambda \,|\, x = 0, \mathcal{P}) &= 1 - e^{-\lambda}
\end{aligned}
$$

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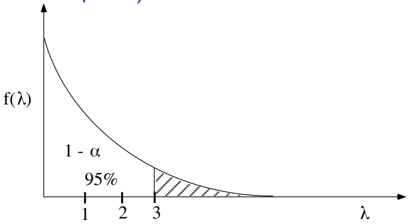

$$f(\lambda\,|\,x=0,\mathcal{P}) \;=\; e^{-\lambda}$$
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Upper probabilistic limit (e.g. at 95% probability):

$$P(\lambda \le \lambda_u\,|\,x=0) = F(\lambda_u\,|\,x=0) \;=\; 0.95$$

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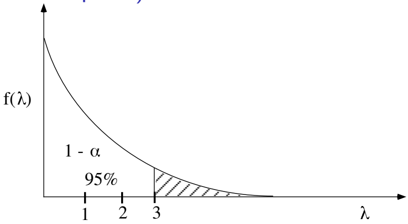

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$$1 - F(\lambda_u\,|\,x=0) = e^{-\lambda_u} = 0.05$$

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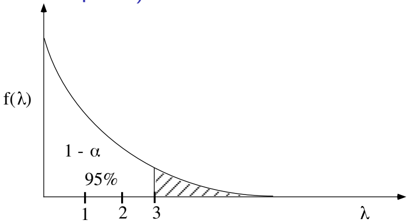

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\lambda_u &= 3 \text{ at } 95\,\% \text{ probability}\,.
\end{aligned}
$$

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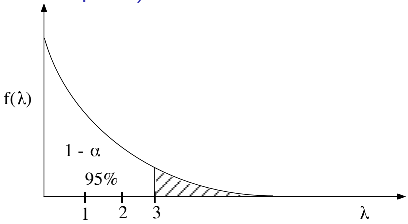

$$\begin{aligned} f(\lambda \,|\, x=0, \mathcal{P}) &= e^{-\lambda} \\ F(\lambda \,|\, x=0, \mathcal{P}) &= 1 - e^{-\lambda} \end{aligned}$$

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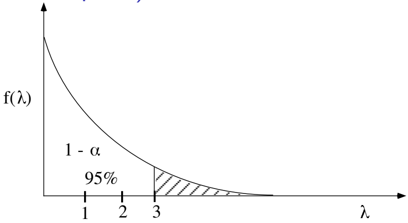

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In this case it works just by chance

$P(A \,|\, B) \leftrightarrow P(B \,|\, A)$

Do you remember? (From first lecture)

In general $P(A \,|\, B) \neq P(B \,|\, A)$

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Everyone was laughing, but this is *more or less* the 'logic' behind frequentistic CL upper/lower bounds

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Very little to laugh...

Inferring Poisson's λ

Conjugate prior

$$f(\lambda \,|\, x) \quad \propto \quad \lambda^x \, e^{-\lambda} \cdot f_\circ(\lambda)$$

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$\Rightarrow$ Gamma distribution

Gamma distribution

$X \sim \text{Gamma}(c, r)$:

$$f(x \,|\, \text{Gamma}(c, r)) = \frac{r^c}{\Gamma(c)} x^{c-1} e^{-r\,x} \qquad \begin{cases} r,\; c > 0 \\ x \geq 0 \end{cases} ,$$

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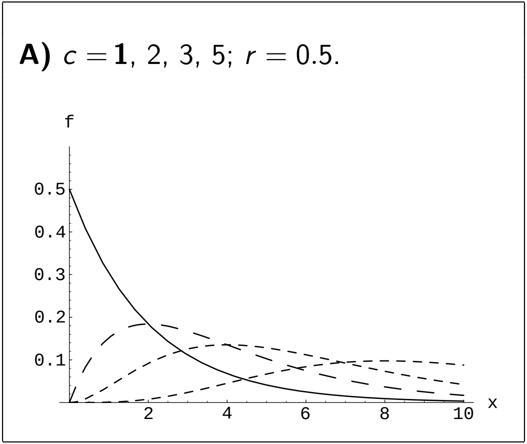
- **The Gamma is a key distribution!**
  The Erlang distribution is important to get a physical intuition of the properties of Gamma and then of the χ^2!

Gamma distribution

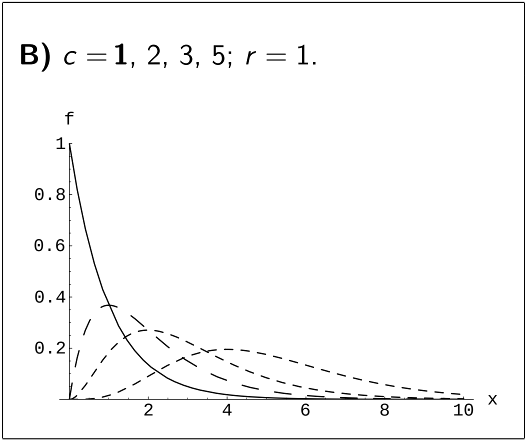
Some examples



r: rate (if the variable is a time, then r is Poisson rate).

Gamma distribution

Some examples



r: rate (rate increases $\rightarrow$ distributions squized)

Gamma distribution

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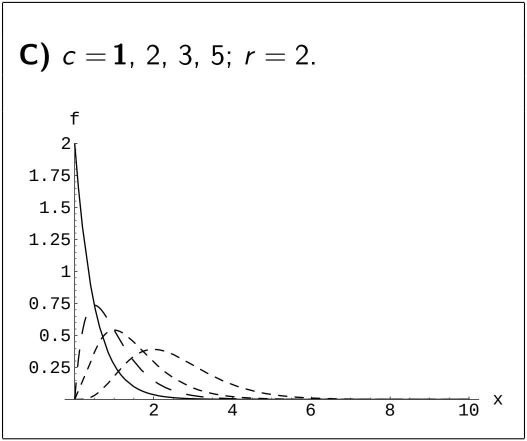

r: rate (rate increases $\rightarrow$ distributions squized)

Gamma (and χ^2) distribution

Summaries

$$
\begin{aligned}
E(X) &= \frac{c}{r} \\
\mathrm{Var}(X) &= \frac{c}{r^2} = \frac{E(X)}{r} \\
\mathrm{mode}(X) &= \begin{cases} 0 & \text{if } c \le 1 \\ \frac{c-1}{r} & \text{if } c > 1 \end{cases}
\end{aligned}
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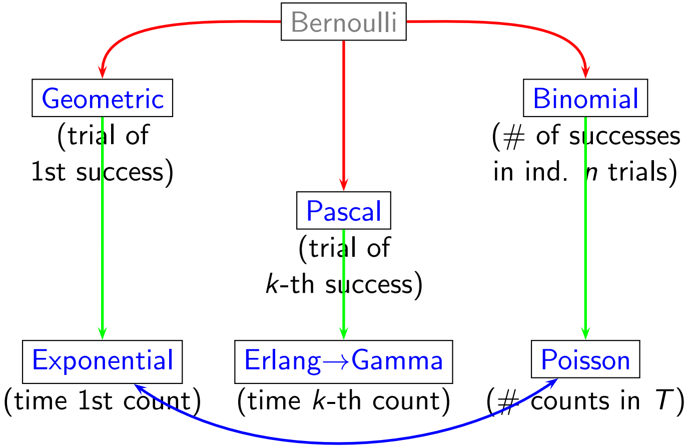
\end{aligned}$$

Therefore, for the χ^2 ($\rightarrow c = \nu/2$, $r = 1/2$)

$$\begin{aligned}
\mathrm{E}(\chi^2) &= \nu \\
\mathrm{Var}(\chi^2) &= 2\,\nu \\
\mathrm{mode}(\chi^2) &= \begin{cases} 0 & \text{if } \nu \leq 2 \\ \nu - 2 & \text{if } \nu > 2 \end{cases}
\end{aligned}$$

Distributions derived from the Bernoulli process



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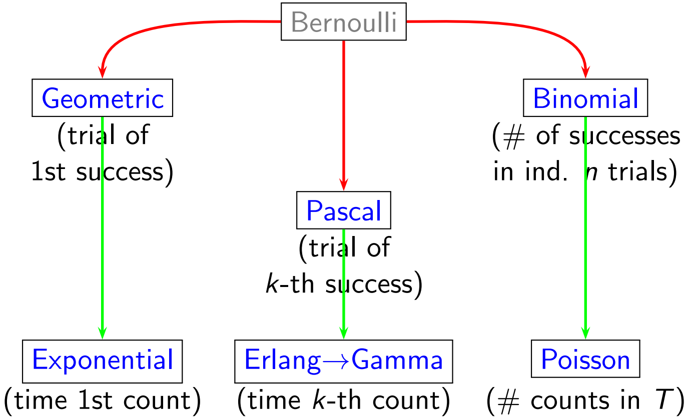

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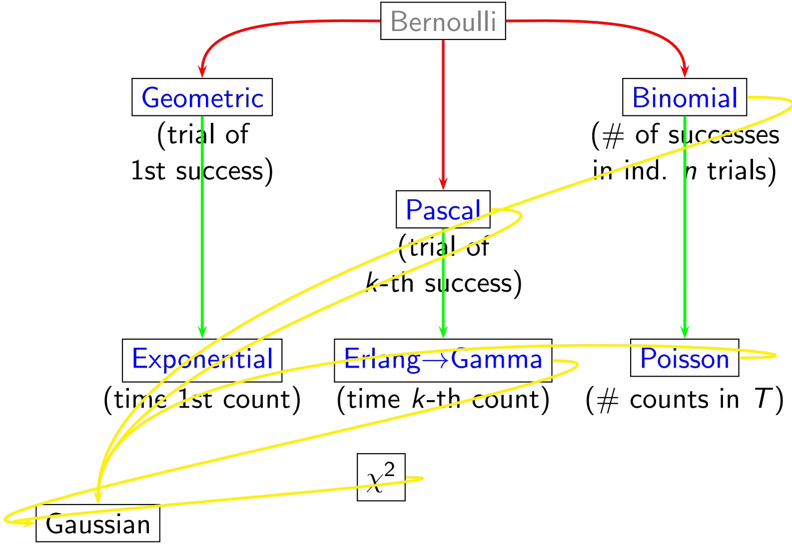

Inferring the Poisson's λ

Use of gamma conjugate prior

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- $c = 1$, $r \to 0$
  $$f(\lambda \,|\, x, \text{Gamma}(c_i = 1, r_i \to 0)) \propto \lambda^x e^{-\lambda}$$

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We have seen how to learn about λ given the observed x (hereafter x_p)

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- by how much? $\rightarrow$ Left as exercise

Example with JAGS

```
# inf_lambda_pred.bug
model {
   X ~ dpois(lambda);
   lambda ~ dexp(0.00001)
   Y ~ dpois(lambda);
}

# inf_lambda_pred.R
library(rjags)
modello = "inf_lambda_pred.bug"  # file con il modello
dati <- NULL  # oggetto con i dati
dati$X  <- 100
jm <- jags.model(modello, dati)  # definisce il modello
update(jm, 100)                  # burn in
catena <- coda.samples(jm, c("lambda","Y"), n.iter=10000)
print(summary(catena))
plot(catena)
```

The End