Measurements, uncertainties and probabilistic inference/forecasting

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Roma, Italy

A simple random walk on a plane

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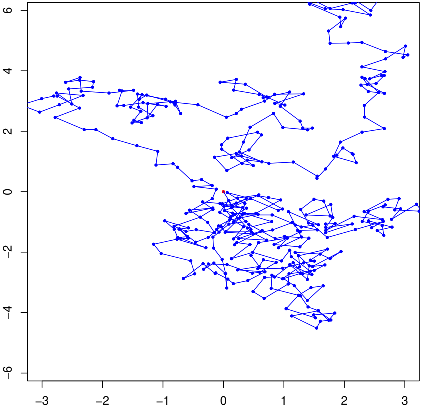

Visiting uniformly the points inside a square

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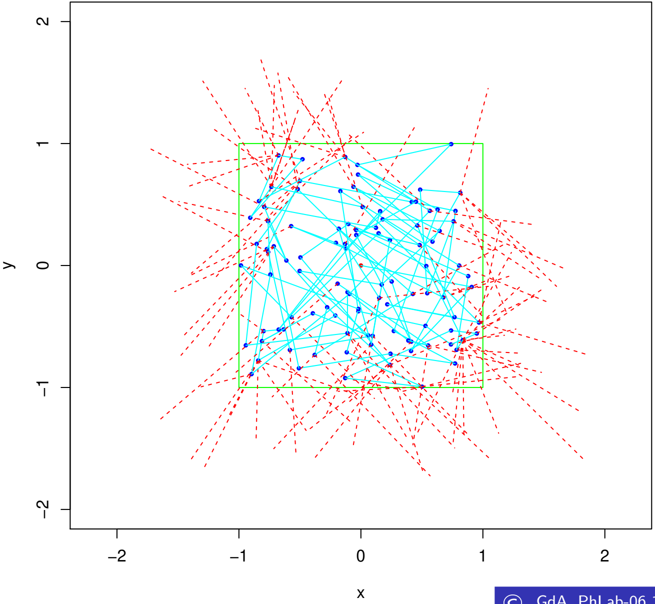
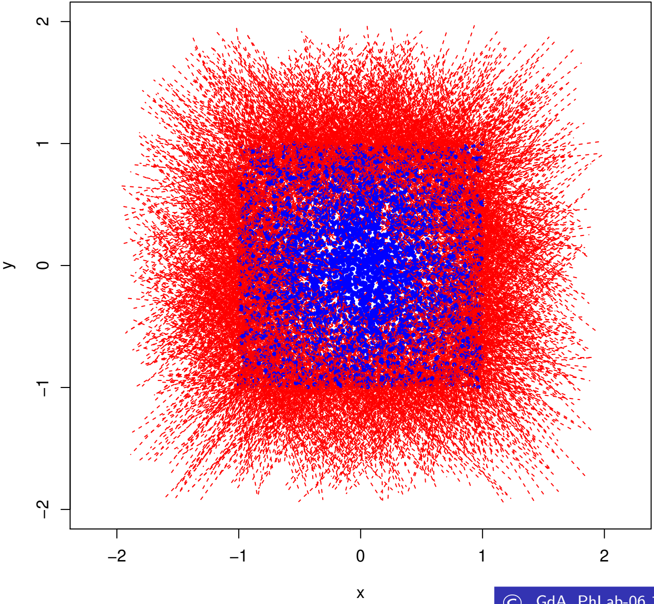
Visiting uniformly the points inside a square

200 attempts



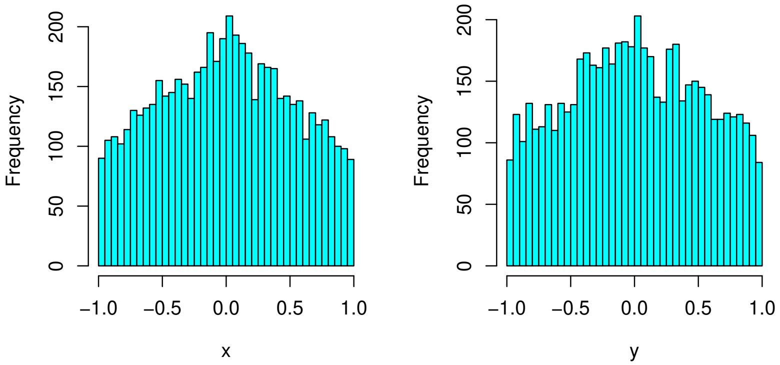
Visiting uniformly the points inside a square

10000 attempts (intermediate cyan 'lines' omitted)



Visiting uniformly the points inside a square

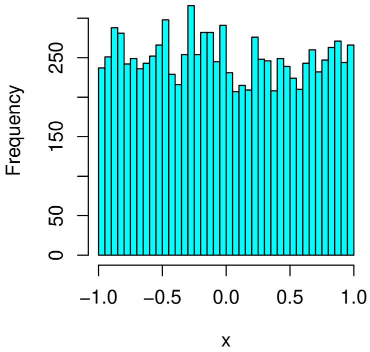
Method 1: count points only once



Not what we wanted!

Visiting uniformly the points inside a square

Method 2: count points several times



$\surd$

Visiting uniformly the points inside a square

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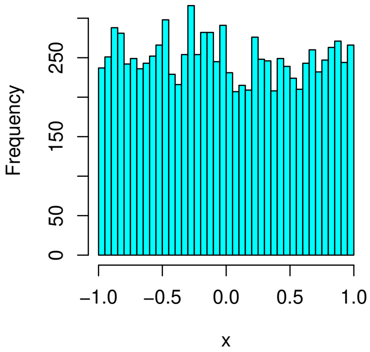

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Why?

Visiting uniformly the points inside a square

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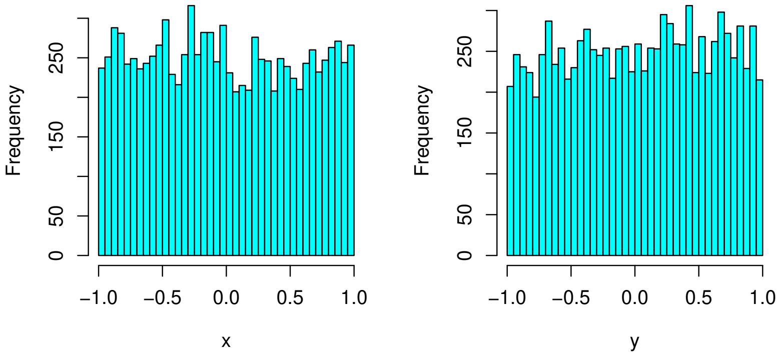

√ **Why?**

MCMC theory left to self study

Inferring μ and σ from n 'independent' measurements

$$f(\mu, \sigma \,|\, \underline{x}, f_0(\mu,\sigma) = k) \;\propto\; \prod_i \frac{1}{\sigma} \exp\left[-\frac{(x_i - \mu)^2}{2\,\sigma^2}\right]$$

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$$\overline{x^2}-\overline{x}^2+\overline{x}^2-2\,\overline{x}\,\mu+\mu^2=\overline{x^2}-\overline{x}^2+(\overline{x}-\mu)^2=s^2+(\overline{x}-\mu)^2$$

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$$f(\mu, \sigma \,|\, \underline{x}, f_0(\mu,\sigma) = k) \quad \propto \quad \sigma^{-n} \exp\left[-\frac{s^2 + (\mu - \overline{x})^2}{2\,\sigma^2/n}\right]$$

with s^2 equal to the sample variance.

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⇒ statistical sufficiency

Inferring μ and σ from n 'independent' measurements

Adding the prediction of x_f

Inferring μ and σ from n 'independent' measurements

Adding the prediction of x_f

$$f(x_f, \mu, \sigma \,|\, \ldots) \;=\; f(x_f \,|\, \mu, \sigma, \ldots) \cdot f(\mu, \sigma \,|\, \ldots)$$

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Try to do it with Metropolis!

Back to $f(\mu, \sigma)$ from a sample

In practice

$$f(\mu, \sigma \,|\, \overline{x}, s, \ldots) \quad \propto \quad \sigma^{-n} \exp\left[-\frac{s^2 + (\mu - \overline{x})^2}{2\,\sigma^2/n}\right] \cdot f_0(\mu, \sigma)$$

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Some remarks are in order:

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Some remarks are in order:

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Some remarks are in order:

- $f(\mu \,|\, \overline{x}, s)$ is in general not Gaussian (**not even starting from a flat prior!**) due to the uncertainty on σ ('convolution over all possible values')
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Back to $f(\mu, \sigma)$ from a sample

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Large sample behaviour starting from uniform priors$^{(*)}$
(with '**std**' for **standard deviation** to avoid confusion with unkown σ)

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$^{(*)}$ The most sensitive is the prior on σ $\Rightarrow$ it induces abstract speculations in mathematicians and statisticians who often have little idea of what they are talking about (**Gauss was Gauss!**).
$\rightarrow$ See **appendix** with details on small numbers

Including uncertainties due to systematics

Exact solution in a special (important) case

Introducing systematics
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(But remember that many 'h's are Physics:
for example you will not be able to get meaningful results on the primordial Universe if you do not first understand the Physics of Cosmic Dust)

Introducing systematics

Several approaches (within probability theory – no adhocheries!)

Uncertainty due to systematic effects is also included in a natural way in this approach. Let us first define the notation (i is the generic index):

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- $\boldsymbol{h} = \{h_1, h_2, \ldots h_{n_h}\}$ is the n-tuple of influence quantities H_i. (see ISO GUM).

Taking into account of uncertain $\boldsymbol{h}$

1. Global inference on $f(\boldsymbol{\mu}, \boldsymbol{h})$

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(*) Usually this is not the case, but in some field of reaserch it is not impossible that e.g. the physical properties of the instrument depend on the physical properties (of materials) you are studying.

Taking into account of uncertain $\boldsymbol{h}$

2. Conditional inference

- Given the observed data, one has a joint distribution of $\boldsymbol{\mu}$ for all possible configurations of $\boldsymbol{h}$:

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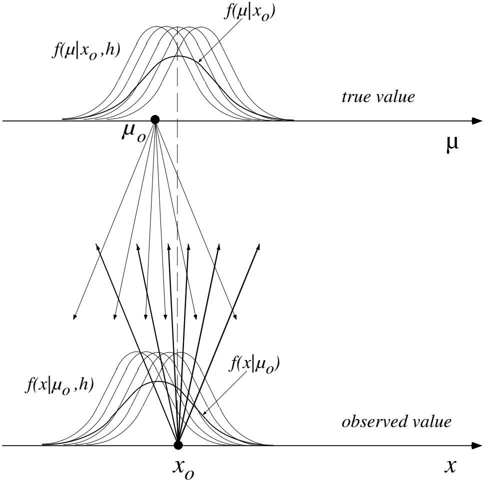
Each conditional result is reweighed with the distribution of beliefs of $\boldsymbol{h}$, using the well-known law of probability:

$$f(\boldsymbol{\mu} \,|\, \boldsymbol{x}) = \int f(\boldsymbol{\mu} \,|\, \boldsymbol{x}, \boldsymbol{h})\, f(\boldsymbol{h})\, \mathrm{d}\boldsymbol{h}\,.$$

Taking into account of uncertain *h*

Conditional inference



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This third approach is particularly convenient to make linear expansions which lead to approximate formulae.

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Model:

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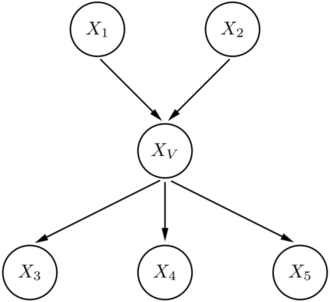
- X is no longer Gaussian distributed around μ, but around $\mu + Z$:

$$X \sim \mathcal{N}(\mu + Z, \sigma)$$

Graphical model

In terms of the generic variables X_i (arXiv:1504.02065)



$$
\begin{aligned}
X_1 &\longleftrightarrow \mu \\
X_2 &\longleftrightarrow Z \\
X_V &\longleftrightarrow \mu + Z \\
X_{3-\ldots} &\longleftrightarrow X_i \text{ (observations)}
\end{aligned}
$$

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Application to the single (equivalent) measuement X_1, with std σ_1

Likelihood:

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Integrating we get

$$f(\mu) = f(\mu\,|\,x_1, \ldots, f_\circ(z)) = \frac{1}{\sqrt{2\,\pi}\,\sqrt{\sigma_1^2+\sigma_Z^2}}\,\exp\left[-\frac{(\mu-x_1)^2}{2\,(\sigma_1^2+\sigma_Z^2)}\right].$$

Systematics due to uncertain offset

Technical remark

It may help to know that

$$\int_{-\infty}^{+\infty} \exp\left[b\,x - \frac{x^2}{a^2}\right] \mathrm{d}x = \sqrt{a^2\,\pi}\,\exp\left[\frac{a^2\,b^2}{4}\right]$$

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- The global standard uncertainty is the quadratic combination of that due to the statistical fluctuation of the data sample and the uncertainty due to the imperfect knowledge of the *systematic effect*:

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- This result (a theorem under well stated conditions!) is often used as a 'prescription', although there are still some "old-fashioned" recipes which require different combinations of the contributions to be performed.

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Measuring μ_1 and μ_2, resulting into x_1 and x_2.

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f(x_1, x_2 \,|\, \mu_1, \mu_2, z) &= \frac{1}{\sqrt{2\,\pi}\,\sigma_1}\, \exp\left[-\frac{(x_1 - \mu_1 - z)^2}{2\,\sigma_1^2}\right] \\
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Systematics due to uncertain offset

Measuring two quantities with the same instrument

Measuring μ_1 and μ_2, resulting into x_1 and x_2.

Setting up the model:

$$
\begin{aligned}
Z &\sim \mathcal{N}(0, \sigma_Z) \\
X_1 &\sim \mathcal{N}(\mu_1 + Z, \sigma_1) \\
X_2 &\sim \mathcal{N}(\mu_2 + Z, \sigma_2) \\
f(x_1, x_2 \,|\, \mu_1, \mu_2, z) &= \frac{1}{\sqrt{2\,\pi}\,\sigma_1}\,\exp\left[-\frac{(x_1-\mu_1-z)^2}{2\,\sigma_1^2}\right] \\
&\quad \times \frac{1}{\sqrt{2\,\pi}\,\sigma_2}\,\exp\left[-\frac{(x_2-\mu_2-z)^2}{2\,\sigma_2^2}\right] \\
&= \frac{1}{2\,\pi\,\sigma_1\sigma_2}\exp\left[-\frac{1}{2}\left(\frac{(x_1-\mu_1-z)^2}{\sigma_1^2} + \frac{(x_2-\mu_2-z)^2}{\sigma_2^2}\right)\right]
\end{aligned}
$$

Systematics due to uncertain offset

Measuring two quantities with the same instrument

$$f(\mu_1, \mu_2 \,|\, x_1, x_2) \;=\; \frac{\int f(x_1, x_2 \,|\, \mu_1, \mu_2, z)\, f_\circ(\mu_1, \mu_2, z)\mathrm{d}z}{\int \ldots \,\mathrm{d}\mu_1\, \mathrm{d}\mu_2\, \mathrm{d}z}$$

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$$f(\mu_1, \mu_2 \,|\, x_1, x_2) = \frac{\int f(x_1, x_2 \,|\, \mu_1, \mu_2, z)\, f_\circ(\mu_1, \mu_2, z)\mathrm{d}z}{\int \ldots\, \mathrm{d}\mu_1\, \mathrm{d}\mu_2\, \mathrm{d}z}$$

$$= \frac{1}{2\,\pi\,\sqrt{\sigma_1^2+\sigma_Z^2}\,\sqrt{\sigma_2^2+\sigma_Z^2}\,\sqrt{1-\rho^2}}$$

$$\times \exp\left\{-\frac{1}{2\,(1-\rho^2)}\left[\frac{(\mu_1-x_1)^2}{\sigma_1^2+\sigma_Z^2}\right.\right.$$

$$\left.\left. -2\,\rho\,\frac{(\mu_1-x_1)(\mu_2-x_2)}{\sqrt{\sigma_1^2+\sigma_Z^2}\,\sqrt{\sigma_2^2+\sigma_Z^2}} + \frac{(\mu_2-x_2)^2}{\sigma_2^2+\sigma_Z^2}\right]\right\}$$

Systematics due to uncertain offset

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\begin{aligned}
f(\mu_1,\mu_2\,|\,x_1,x_2) &= \frac{\int f(x_1,x_2\,|\,\mu_1,\mu_2,z)\,f_\circ(\mu_1,\mu_2,z)\mathrm{d}z}{\int \ldots\,\mathrm{d}\mu_1\,\mathrm{d}\mu_2\,\mathrm{d}z} \\
&= \frac{1}{2\,\pi\,\sqrt{\sigma_1^2+\sigma_Z^2}\,\sqrt{\sigma_2^2+\sigma_Z^2}\,\sqrt{1-\rho^2}} \\
&\quad \times \exp\left\{-\frac{1}{2\,(1-\rho^2)}\left[\frac{(\mu_1-x_1)^2}{\sigma_1^2+\sigma_Z^2}\right.\right. \\
&\quad \left.\left. -2\,\rho\,\frac{(\mu_1-x_1)(\mu_2-x_2)}{\sqrt{\sigma_1^2+\sigma_Z^2}\,\sqrt{\sigma_2^2+\sigma_Z^2}} + \frac{(\mu_2-x_2)^2}{\sigma_2^2+\sigma_Z^2}\right]\right\}
\end{aligned}
$$

where

$$
\rho = \frac{\sigma_Z^2}{\sqrt{\sigma_1^2+\sigma_Z^2}\,\sqrt{\sigma_2^2+\sigma_Z^2}}\,.
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\end{aligned}
$$

where

$$
\rho = \frac{\sigma_Z^2}{\sqrt{\sigma_1^2+\sigma_Z^2}\,\sqrt{\sigma_2^2+\sigma_Z^2}}\,.
$$

$\Rightarrow$ bivariate normal distribution!

Systematics due to uncertain offset

Summary:

$$\mu_1 \sim \mathcal{N}\left(x_1, \sqrt{\sigma_1^2 + \sigma_Z^2}\right)$$

$$\mu_2 \sim \mathcal{N}\left(x_2, \sqrt{\sigma_2^2 + \sigma_Z^2}\right)$$

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\rho &= \frac{\sigma_Z^2}{\sqrt{\sigma_1^2 + \sigma_Z^2}\,\sqrt{\sigma_2^2 + \sigma_Z^2}} \\
\mathrm{Cov}(\mu_1, \mu_2) &= \rho\,\sigma_{\mu_1}\sigma_{\mu_2} \\
&= \rho\,\sqrt{\sigma_1^2 + \sigma_Z^2}\,\sqrt{\sigma_2^2 + \sigma_Z^2} = \sigma_Z^2
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Checks, defining $S = \mu_1 + \mu_2$ and $D = \mu_1 - \mu_2$

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$$D \sim \mathcal{N}\left(x_1 - x_2, \sqrt{\sigma_1^2 + \sigma_2^2}\right)$$

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$$
\begin{aligned}
D &\sim \mathcal{N}\left(x_1-x_2, \sqrt{\sigma_1^2+\sigma_2^2}\right)\\
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As more or less intuitively expected from an offset!

A simple example with JAGS

```
model {
   for (i in 1:length(x)) {
      x[i] ~ dnorm(mu.s, tau)
   }
   mu.s <- mu + z;
   mu ~ dnorm(0.0, 1.0E-6)
   z ~ dnorm(0, 1/sigma.z^2)
   tau ~ dgamma(1.0, 1.0E-4)
   sigma <- 1/sqrt(tau)
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}
```

=> `norm_sist_z.R`

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- extend it to two data sets measuring μ_1 and μ_2 with common systematics;
- include also derived quantities, like sum and difference;
- modify the model in order to describe a systematics affetting the scale: $f = 1 \pm \sigma_f$;
- then, add other derived quantities, like product and ratio.

Fits – introduction

- In a probabilistic framework the issue of the fits is nothing but

  parametric inference.

- set up the model,
  e.g. $\mu_{y_i} = m\,\mu_{x_i} + c$

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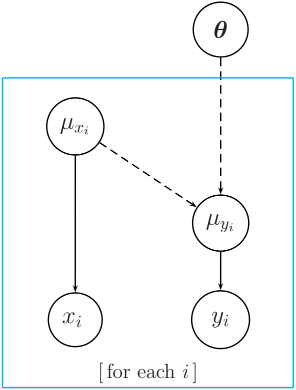

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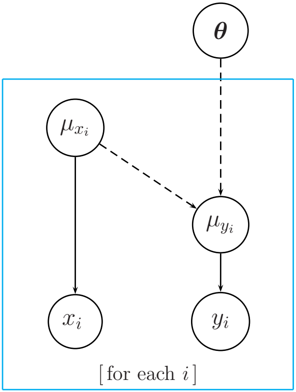

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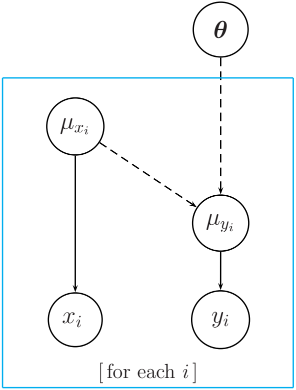

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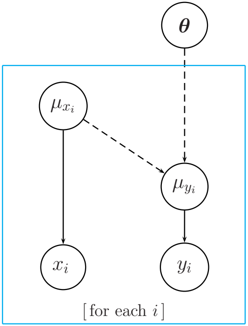
$\rightarrow f(\boldsymbol{\theta}\,|\,\mathrm{x}, \mathrm{y}, I)$
$\rightarrow f(m, c\,|\,\mathrm{x}, \mathrm{y}, \boldsymbol{\sigma})$, in the case of case of linear fit with "σ's known a priori" (!)

Linear fit – introduction



- Deterministic links between μ_x's and μ_y's.

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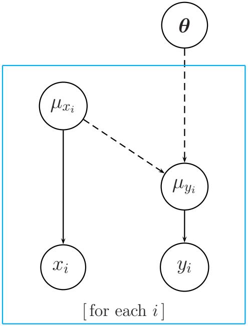

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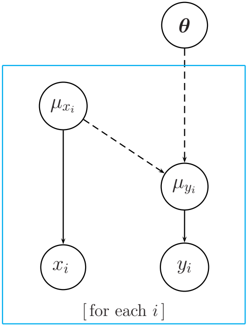

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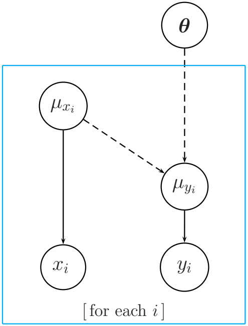

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- $\Rightarrow$ aim of fit (<u>σ's known</u>): $\{\boldsymbol{x}, \boldsymbol{y}\} \rightarrow \boldsymbol{\theta} = (m, c)$
- If σ_x's and σ_y's are unkown and assumed all equal $\{\boldsymbol{x}, \boldsymbol{y}\} \rightarrow \boldsymbol{\theta} = (m, c, \sigma_x, \sigma_y)$
- etc...

Linear fit – simplest case

$$f(m, c \,|\, \boldsymbol{x}, \boldsymbol{y}, I) \;\;\propto\;\; f(\boldsymbol{x}, \boldsymbol{y} \,|\, m, c, I) \cdot f_0(m, c)$$

Simplifying hypotheses:

- No error on $\mu_x \Rightarrow \mu_{x_i} = x_i$:
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$$\begin{aligned} f(m, c \,|\, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\sigma}) &\propto \exp\left[-\sum_i \frac{(y_i - \mu_{y_i})^2}{2\,\sigma_i^2}\right] \cdot f_0(m, c) \\ &\propto \exp\left[-\frac{1}{2}\sum_i \frac{(y_i - m\,x_i - c)^2}{\sigma_i^2}\right] \cdot f_0(m, c) \end{aligned}$$

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$$\;\;\propto\;\; \exp\left[-\frac{1}{2}\sum_i \frac{(y_i - m\, x_i - c)^2}{\sigma_i^2}\right] \cdot f_0(m, c)$$

$\Rightarrow$ flat priors: inference only depends on $\exp\left[-\frac{1}{2}\sum_i \frac{(y_i - m\, x_i - c)^2}{\sigma_i^2}\right]$.

Least squares and 'Gaussian tricks' on linear fits

$$f(m, c \,|\, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\sigma}) \;\propto\; \exp\left[-\frac{\sum_i (y_i - m\,x_i - c)^2}{2\,\sigma_i^2}\right] \cdot f_0(m, c)$$

- <u>If</u> the prior is irrelevant and the σ's are all equal, than the maximum of the posterior is obtained when the sum of the squares is minimized:
  $\Rightarrow$ Least Square 'Principle'.

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- You might recognize at the exponent: $\chi^2/2$:
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  $\Rightarrow$ $\varphi(m, c) \propto \chi^2$.

$\Rightarrow$ same result of the detailed one is achieved, simply because the problem is linear!
(No garantee in general!)

Uncertain standard deviation

In the probabilistic approach it is rather simple: just add σ in $\boldsymbol{\theta}$ to infer.

- For example, if we have good reasons to belief that the σ's are all equal, then

$$f(m, c, \sigma \,|\, \boldsymbol{x}, \boldsymbol{y}) \;\propto\; \sigma^{-n} \exp\left[-\frac{\sum_i (y_i - m\,x_i - c)^2}{2\,\sigma^2}\right] \cdot f_0(m, c, \sigma)$$

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Residuals? Ok if there are many points, otherwise we do not take into account the uncertainty on σ and its effect on the probability function of m and c.

Note: as long as σ is constant (although unknown) and the prior flat in m and c the best estimates of m and c do not depend in σ.

Linear fits with uncertain σ in JAGS

Model

```
var mu.y[N];
model{
    for (i in 1:N) {
      y[i] ~ dnorm(mu.y[i], tau);
      mu.y[i] <- x[i]*m + c;
    }
    c ~ dnorm(0, 1.0E-6);
    m ~ dnorm(0, 1.0E-6);
    tau ~ dgamma(1.0, 1.0E-6);
    sigma <- 1.0/sqrt(tau);
}
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    sigma <- 1.0/sqrt(tau);
}
```

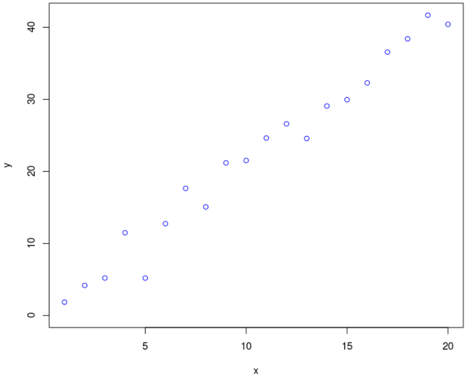
Simulated data

```
m.true = 2; c.true = 1; sigma.true=2
x = 1:20
y = m.true * x + c.true + rnorm(length(x), 0, sigma.true)

plot(x,y, col='blue',ylim=c(0,max(y)) )
```

Linear fits with uncertain σ in JAGS

Plot of simulated data



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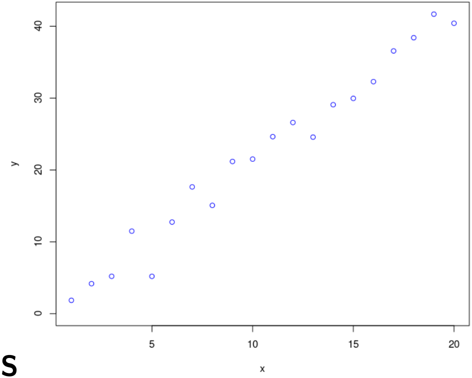


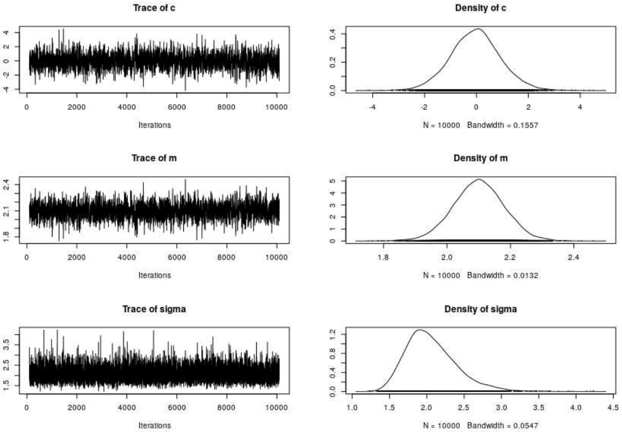
Calling JAGS

```
ns=10000
jm <- jags.model(model, data, inits)
update(jm, 100)
chain <- coda.samples(jm, c("c","m","sigma"), n.iter=ns)
```

Linear fits with uncertain σ in JAGS

$\Rightarrow$ `linear_fit.R`

JAGS summary



$c = -0.04 \pm 0.96$; $m = 2.10 \pm 0.08$; $\sigma = 2.06 \pm 0.34$

Linear fits with uncertain σ in JAGS

‘Check’ the result

```
c     <- as.vector(chain[[1]][,1])
m     <- as.vector(chain[[1]][,2])
sigma <- as.vector(chain[[1]][,3])
plot(x,y, col='blue',ylim=c(0,max(y)) )
abline(mean(c), mean(m), col='red')
```

Linear fits with uncertain σ in JAGS

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c     <- as.vector(chain[[1]][,1])
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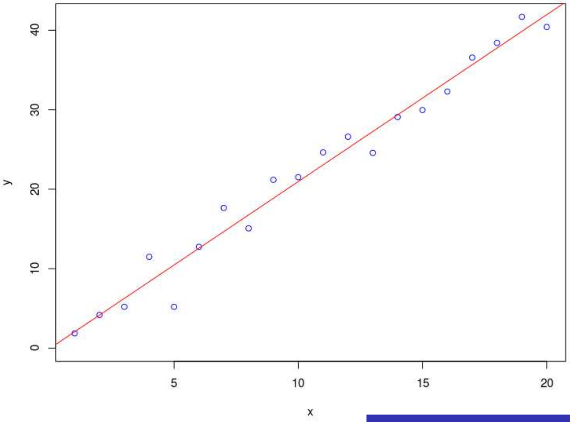

Linear fits with uncertain σ in JAGS

Correlation between m and c

```
plot(m,c,col='cyan')
cat(sprintf("rho(m,x) = %.3f\n", cor(m,c) ))
```

Linear fits with uncertain σ in JAGS

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plot(m,c,col='cyan')
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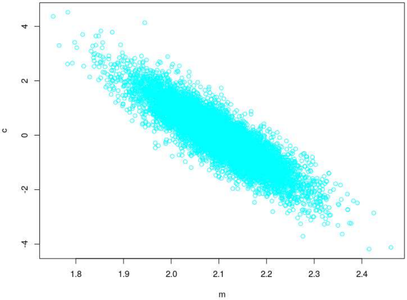

$$\rho(m, c) = -0.88$$

Linear fits with uncertain σ in JAGS

Check with R `lm()` (least square)

```
plot(x,y, col='blue',ylim=c(0,max(y)) )
abline(mean(c), mean(m), col='red') # JAGS
abline(lm(y~x), col='black')        # least squares
```

Linear fits with uncertain σ in JAGS

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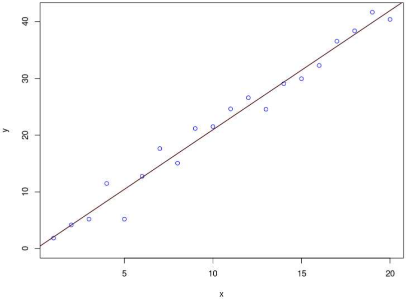

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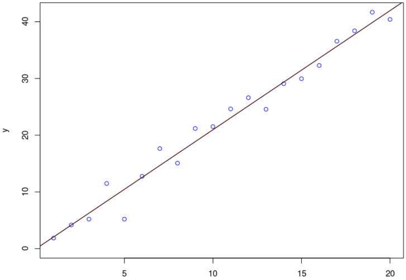

Linear model line ($c = -0.05$, $m = 2.10$) covers perfectly the JAGS result

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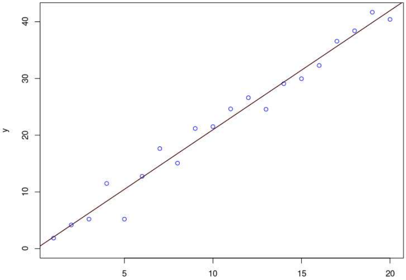

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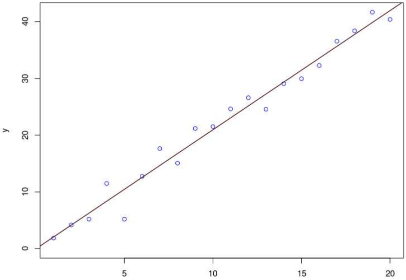

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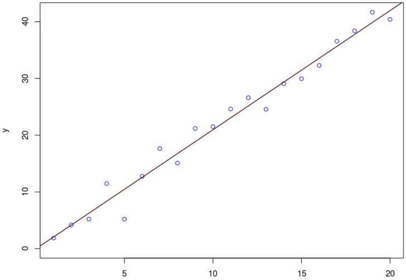

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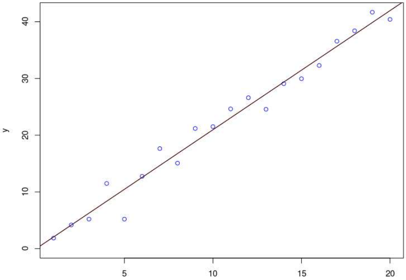

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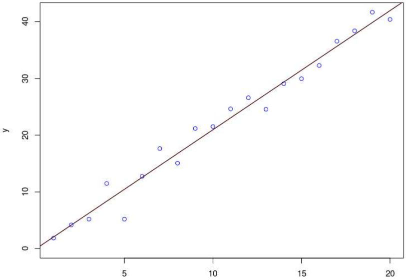

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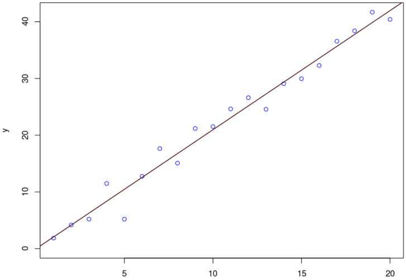

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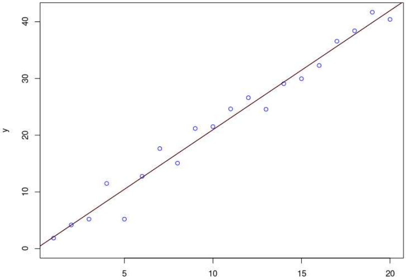

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Otherwise: $\Rightarrow f(c, m, \sigma \,|\, \text{data points})$

Forecasting new μ_y and new y

Imagine we are interested at "y at $x_f = 30$" (referring to our 'data').

Forecasting new μ_y and new y

Imagine we are interested at "y at $x_f = 30$" (referring to our 'data').

- First at all it is important to distinguish

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\mu_y(x_f) &\;\rightarrow\; \mu_y(\mu_{x_f}) \qquad \text{(no error on } x\text{)} \\
y(x_f) &\;\rightarrow\; y(\mu_{x_f})
\end{aligned}
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Our problem

$$f(\mu_{y_f}\,|\,\text{data}, x_f) = \int f(\mu_{y_f}\,|\,m, c, x_f) \cdot f(m, c\,|\,\text{data})\,\mathrm{d}c\,\mathrm{d}m$$

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Forecasting new μ_y and new y

Including prediction in the JAGS model

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model{
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      mu.y[i] <- x[i] * m + c;
    }
    mu.yf <- xf * m + c;      # future 'true value' for x=xf
    yf ~ dnorm(mu.yf, tau);   # future 'observation for x=xf
    c ~ dnorm(0, 1.0E-6);
    m ~ dnorm(0, 1.0E-6);
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    sigma <- 1.0/sqrt(tau);
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Or we can do the 'integral' by sampling, using the MCMC histories of the quantities of interest
(see previous model, without prediction)

Forecasting new μ_y and new y

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Or we can do the 'integral' by sampling, using the MCMC histories of the quantities of interest
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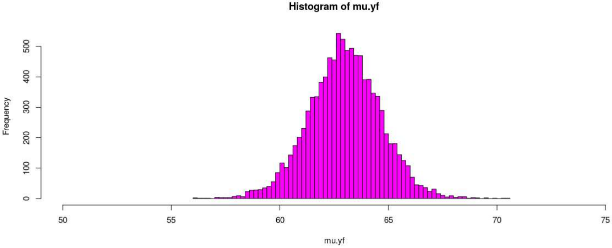
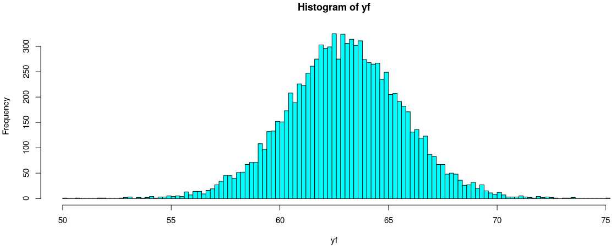
$\Rightarrow$ Left as exercise

Forecasting new μ_y and new y with JAGS





$\mu_y(x = 30) = 63.0 \pm 1.7$; $\quad y(x = 30) = 63.0 \pm 2.7$

Try with Root ;-) ['data' on the web site]

The End

Appendix on small samples

Inferring μ and σ from a sample

(Gaussian, independent observations)

$$f(\mu, \sigma \,|\, \underline{x}) \quad \propto \quad \sigma^{-n} \exp\left[-\frac{\overline{x^2} - 2\,\mu\,\overline{x} + \mu^2}{2\,\sigma^2/n}\right] \cdot f_0(\mu, \sigma)$$

Inferring μ and σ from a sample

(Gaussian, independent observations)

$$
\begin{aligned}
f(\mu,\sigma\,|\,\underline{x}) &\propto \sigma^{-n}\exp\left[-\frac{\overline{x^2}-2\,\mu\,\overline{x}+\mu^2}{2\,\sigma^2/n}\right]\cdot f_0(\mu,\sigma)\\
&\propto \sigma^{-n}\exp\left[-\frac{\overline{x^2}-\overline{x}^2+\overline{x}^2-2\,\mu\,\overline{x}+\mu^2}{2\,\sigma^2/n}\right]\cdot f_0(\mu,\sigma)
\end{aligned}
$$

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&\propto \sigma^{-n} \exp\left[-\frac{\overline{x^2} - \overline{x}^2 + \overline{x}^2 - 2\,\mu\,\overline{x} + \mu^2}{2\,\sigma^2/n}\right] \cdot f_0(\mu, \sigma) \\
&\propto \sigma^{-n} \exp\left[-\frac{s^2 + (\mu - \overline{x})^2}{2\,\sigma^2/n}\right] \cdot f_0(\mu, \sigma)
\end{aligned}
$$

with $s^2 = \overline{x^2} - \overline{x}^2$, variance of the sample.

Inferring μ and σ from a sample

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$$f(\mu,\sigma\,|\,\underline{x}) \;\propto\; \sigma^{-n}\exp\left[-\frac{\overline{x^2}-2\,\mu\,\overline{x}+\mu^2}{2\,\sigma^2/n}\right]\cdot f_0(\mu,\sigma)$$

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with $s^2=\overline{x^2}-\overline{x}^2$, variance of the sample.

- the inference on μ and σ depends only on s^2 and $\overline{x}$ (and on the priors, as it has to be!).

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$$f(\mu,\sigma\,|\,\underline{x}) \;\propto\; \sigma^{-n}\exp\left[-\frac{\overline{x^2}-2\,\mu\,\overline{x}+\mu^2}{2\,\sigma^2/n}\right]\cdot f_0(\mu,\sigma)$$

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- the inference on μ and σ depends only on s^2 and $\overline{x}$ (and on the priors, as it has to be!).
- Evaluate $f(\mu,\sigma\,|\,\overline{x},s)$ and then

$$f(\mu\,|\,\overline{x},s) \;=\; \int_0^\infty f(\mu,\sigma\,|\,\overline{x},s)\,d\sigma$$

Inferring μ and σ from a sample

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f(\mu,\sigma\,|\,\underline{x}) &\propto \sigma^{-n}\exp\left[-\frac{\overline{x^2}-2\,\mu\,\overline{x}+\mu^2}{2\,\sigma^2/n}\right]\cdot f_0(\mu,\sigma)\\
&\propto \sigma^{-n}\exp\left[-\frac{\overline{x^2}-\overline{x}^2+\overline{x}^2-2\,\mu\,\overline{x}+\mu^2}{2\,\sigma^2/n}\right]\cdot f_0(\mu,\sigma)\\
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- Evaluate $f(\mu,\sigma\,|\,\overline{x},s)$ and then

$$
\begin{aligned}
f(\mu\,|\,\overline{x},s) &= \int_0^{\infty} f(\mu,\sigma\,|\,\overline{x},s)\,d\sigma\\
f(\sigma\,|\,\overline{x},s) &= \int_{-\infty}^{+\infty} f(\mu,\sigma\,|\,\overline{x},s)\,d\mu
\end{aligned}
$$

Inferring μ and σ from a sample

(Gaussian, independent observations – prior uniform on σ)

$$f(\mu, \sigma \,|\, \underline{x}) \;\; \propto \;\; \sigma^{-n} \exp\left[-\frac{s^2 + (\mu - \overline{x})^2}{2\,\sigma^2/n}\right]$$

Marginalizing[1]

$$f(\mu \,|\, \underline{x}) \;\; = \;\; \int_0^{\infty} f(\mu, \sigma \,|\, \underline{x})\, \mathrm{d}\sigma$$

[1]The integral of interest is

$$\int_0^{\infty} z^{-n} \exp\left[-\frac{c}{2\,z^2}\right] \mathrm{d}z = 2^{(n-3)/2}\, \Gamma\left[\frac{1}{2}(n-1)\right] c^{-(n-1)/2}.$$

Inferring μ and σ from a sample

(Gaussian, independent observations – prior uniform on σ)

$$f(\mu, \sigma \,|\, \underline{x}) \;\propto\; \sigma^{-n} \exp\left[-\frac{s^2 + (\mu - \overline{x})^2}{2\,\sigma^2/n}\right]$$

Marginalizing[1]

$$\begin{aligned} f(\mu \,|\, \underline{x}) &= \int_0^\infty f(\mu, \sigma \,|\, \underline{x})\, \mathrm{d}\sigma \\ &\propto \left((\overline{x} - \mu)^2 + s^2\right)^{-(n-1)/2} \end{aligned}$$

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f(\mu \,|\, \underline{x}) &\propto \left(1 + \frac{(\mu - \overline{x})^2}{s^2}\right)^{-(n-1)/2} \quad ?? \\
&\propto \left(1 + \frac{(\mu - \overline{x})^2}{(n-2)\, s^2/(n-2)}\right)^{-((n-2)+1)/2}
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with

$$
\begin{aligned}
\nu &= n - 2 \\
t &= \frac{\mu - \overline{x}}{s/\sqrt{n-2}}\,,
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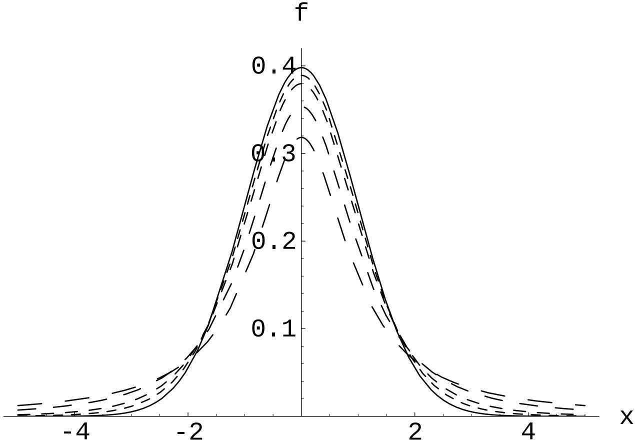
that is

$$
\mu = \overline{x} + \frac{s}{\sqrt{n-2}}\,t\,,
$$

where t is a "Student t" with $\nu = n-2$

Student t



Examples of Student t for ν equal to 1 , 2, 5, 10 and 100 ($\approx$ "∞").

Inferring μ and σ from a sample

(Gaussian, independent observations – prior uniform on μ and σ)

In summary,

$$\frac{\mu - \overline{x}}{s/\sqrt{n-2}} \quad \sim \quad \text{Student}(\nu = n-2)$$

Inferring μ and σ from a sample

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In summary,

$$\begin{aligned}\frac{\mu-\overline{x}}{s/\sqrt{n-2}} &\sim \text{Student}(\nu=n-2)\\ \text{E}(\mu) &\stackrel{(n>3)}{=} \overline{x}\\ \sigma(\mu) &\stackrel{(n>4)}{=} \frac{s}{\sqrt{n-4}}\,.\end{aligned}$$

The uncertainty on σ increases the probability of the values of μ far from $\overline{x}$:

- not only the standard uncertainty increases, but the distribution itself changes and, as 'well know' the t distribution has 'higher' tails.

Inferring μ and σ from a sample

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However, when n is very large the Gaussian distribution is recovered (the t-distribution tends to a gaussian), with $\sigma(\mu) = s/\sqrt{n}$.

Expected value and variance only exist above certain values of n:

$$\begin{aligned} \mathrm{E}(\mu) &\overset{(n>3)}{=} \overline{x} \\ \sigma(\mu) &\overset{(n>4)}{=} \frac{s}{\sqrt{n-4}}\,. \end{aligned}$$

[2] Flat priors are good for teaching purposes, but when the result hurts with our beliefs it means we have to use priors that match with previous knowledge.

Inferring μ and σ from a sample

Misunderstandings and 'myths' related to the Student t distribution

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So what?

It is just a reflex of the fact that we have used, for lazyness,[2] priors which are indeed absurd.

- In no measurement we beleive that μ and/or σ could be 'infinite'.
- Just plug in some reasonable, although very vagues, proper priors, and the problem disappears.

---

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Inferring μ and σ from a sample

(Gaussian, independent observations – prior uniform on μ and σ)

- Large n limit:

$$
\begin{aligned}
\mathrm{E}(\mu) &\xrightarrow{n\to\infty} \overline{x} \\
\sigma(\mu) &\xrightarrow{n\to\infty} \frac{s}{\sqrt{n}} \\
\mu &\xrightarrow{n\to\infty} \sim \mathcal{N}(\overline{x}, \frac{s}{\sqrt{n}}).
\end{aligned}
$$

Inferring μ and σ from a sample

(Gaussian, independent observations – prior uniform on μ and σ)

Marginal $f(\sigma)$

$$f(\sigma\,|\,\overline{x},s) \;=\; \int_{-\infty}^{+\infty} f(\mu,\sigma\,|\,\overline{x},s)\,d\mu$$

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\begin{aligned}
f(\sigma\,|\,\overline{x}, s) &= \int_{-\infty}^{+\infty} f(\mu, \sigma\,|\,\overline{x}, s)\, d\mu \\
&\propto \sigma^{-n} \exp\left[-\frac{n\, s^2}{2\,\sigma^2}\right] \int_{-\infty}^{+\infty} \exp\left[-\frac{n\,(\overline{x}-\mu)^2}{2\sigma^2}\right] \mathrm{d}\mu
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That is. . . (no special function)

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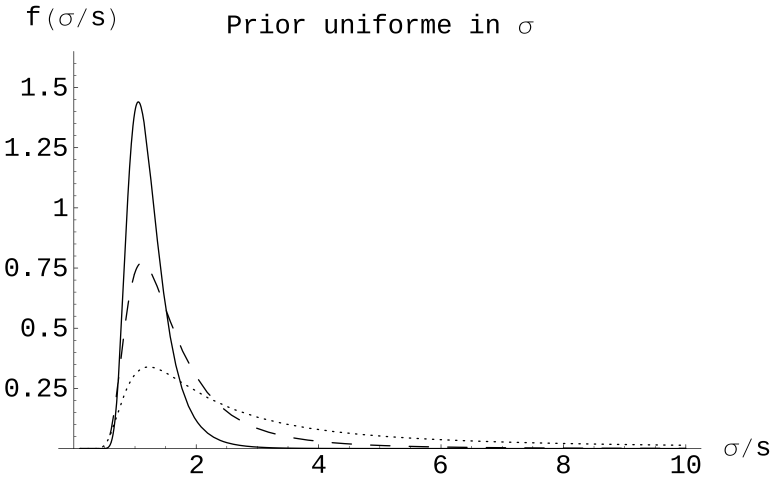
That is... (no special function)

[But if we would use $\tau = 1/\sigma^2$ we would recognize a Gamma...]

Inferring μ and σ from a sample

(Gaussian, independent observations – prior uniform on *mu* and σ)



$n = 3$ (dotted), $n = 5$ (dashed) e $n = 10$ (continous).

Inferring μ and σ from a sample

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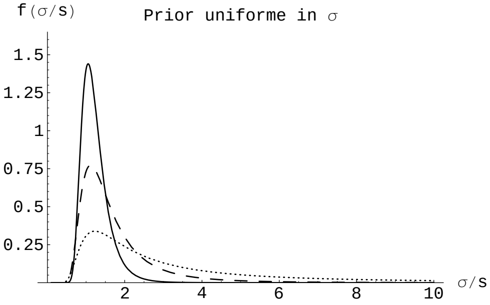

$$\mathrm{E}(\sigma) \xrightarrow[n\to\infty]{} s$$

$$\mathrm{std}(\sigma) \xrightarrow[n\to\infty]{} \frac{s}{\sqrt{2\,n}}$$

$$\sigma \xrightarrow[n\to\infty]{} \sim \mathcal{N}(s, \frac{s}{\sqrt{2\,n}})\,.$$

Inferring μ and σ from a sample

(Gaussian, independent observations – prior uniform on μ and σ)

Using the "Gaussian trick"

$$\varphi(\mu,\sigma) \quad = \quad n \ln\sigma + \frac{s^2 + (\mu - \overline{x})^2)}{2\,\sigma/n}$$

Inferring μ and σ from a sample

(Gaussian, independent observations – prior uniform on μ and σ)

Using the "Gaussian trick"

$$\varphi(\mu,\sigma) \;=\; n\ln\sigma + \frac{s^2 + (\mu-\overline{x})^2)}{2\,\sigma/n}$$

First derivatives:

$$\frac{\partial\varphi}{\partial\mu} \;=\; \frac{\mu-\overline{x}}{\sigma/n}$$

$$\frac{\partial\varphi}{\partial\sigma} \;=\; \frac{n}{\sigma} - \frac{n\,s^2}{\sigma^3}$$

Inferring μ and σ from a sample

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From which it follows (equating the derivatives to zero)

$$\begin{aligned} \mathrm{E}(\mu) &= \overline{x} \\ \mathrm{E}(\sigma) &= s \end{aligned}$$

(They are indeed the modes!)

Inferring μ and σ from a sample

(Gaussian, independent observations – prior uniform on μ and σ)

Hessian calculated at $\mu = \overline{x}$ and $\sigma = s$ (hereafter 'm'):

$$\left.\frac{\partial^2\varphi}{\partial\mu^2}\right|_m = \left.\frac{n}{\sigma^2}\right|_m = \frac{n}{s^2}$$

$$\left.\frac{\partial^2\varphi}{\partial\sigma^2}\right|_m = \left.\left(-\frac{n}{\sigma^2} + \frac{3\,(s^2 + (\mu-\overline{x})^2)}{\sigma^4/n}\right)\right|_m = \frac{2\,n}{s^2}$$

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\end{aligned}$$

It follows

$$\begin{aligned}
\mathrm{std}(\mu) &= \frac{s}{\sqrt{n}} \\
\mathrm{std}(\sigma) &= \frac{s}{\sqrt{2\,n}}\,,
\end{aligned}$$

reobtaining the large number limit.

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$$\mathrm{std}(\mu) = \frac{s}{\sqrt{n}}$$

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reobtaining the large number limit. And, notice, $\rho(\mu, \sigma) = 0\,.$

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reobtaining the large number limit. And, notice, $\rho(\mu, \sigma) = 0$.

Q.: Are they independent?

Inferring μ and σ from a sample

(Gaussian, independent observations. Expression the Gaussian in terms of $\tau = 1/\sigma^2$)

$$f(\mu, \sigma \,|\, \underline{x}) \;\propto\; \sigma^{-n} \exp\left[-\frac{s^2 + (\mu - \overline{x})^2}{2\,\sigma^2/n}\right] \cdot f_0(\mu, \sigma)$$

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It is technically convenient to use $\tau = 1/\sigma^2$:

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For a fixed μ (and observed s and $\overline{x}$)

$$f(\tau \,|\, \underline{x}, \mu) \;\; \propto \;\; \tau^{\alpha}\, e^{-\beta\,\tau} \cdot f_0(\tau)$$

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Do you recongnize a *famous* mathematical form?

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$\Rightarrow$ Gibbs sampling

Practical introduction to BUGS

- Introducing the *bug* language to build up the models.
- Running the model (including data and 'inits') in the OpenBUGS GUI.
- Analysing the resulting chain in R.