The Fermi's Bayes Theorem

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Enrico Fermi is usually associated by the general public with the first self-staining nuclear chain reaction and, somehow, with the Manhattan Project to build the first atomic bomb. But besides these achievements, that have set a mark in the history, his contribution to physics - and especially fundamental physics - has been immense, as testified for example by the frequency with which his name, or a derived noun or adjective, appears in the scientific literature (fermi, fermium, fermion, F. interaction, F. constant, Thomas-F. model, F. gas, F. energy, F. coordinates, F. acceleration mechanism, etc.). Indeed he has been one of the founding fathers of atomic, nuclear, particle and solid state physics, with some relevant contributions even in general relativity and astrophysics.

He certainly mastered probability theory and one of his chief interests through his life was the study of the statistical behavior of physical systems of free or interacting particles. Indeed, there is a 'statistics' that carries his name, together with that of the co-inventor Paul Dirac, and the particles described by the Fermi-Dirac statistics are called fermions.

Among the several other contributions of Enrico Fermi to statistical mechanics, perhaps the most important one is contained in his last paper, written with John Pasta and Stan Ulam. Without entering into the physics contents of the paper (it deals with what is presently known as the 'FPU problem') it is worth mentioning the innovative technical-methodological issue of the work: the time evolution of a statistical system (just a chain of nonlinearly coupled masses and springs) was simulated by computer. The highly unexpected result stressed the importance of using numerical simulations as a research tool complementary to theoretical studies or laboratory experiments. Therefore, Fermi, who was unique in mastering at his level both theory and experiments, was also one of the first physicists doing 'computer experiments'.

In fact, with the advent of the first electronic computers, Fermi immediately realized the importance of using them to solve complex problems that lead to difficult or intractable systems of integral-differential equations. One use of the computer consisted in discretizing the problem and solving it by numerical steps (as in the FPU problem). The other use consisted in applying sampling techniques, of which Fermi is also recognized to be a pioneer. It seems in fact, as also acknowledged by Nick

Metropolis (`http://library.lanl.gov/cgi-bin/getfile?00326866.pdf`), that Fermi contrived and used the Monte Carlo method to solve practical neutron diffusion problems in the early nineteen thirties, i.e. fifteen years before the method was finally 'invented' by Ulam, named by Metropolis, and implemented on the first electronic computer thanks to the interest and drive of Ulam and John von Neumann.

After this short presentation of the character, with emphasis on something that might concern the reader of this bulletin, one might be interested about Fermi and 'statistics', meant as a data analysis tool. During my studies and later I had never found Fermi's name in the books and lecture notes on statistics I was familiar with. It has then been a surprise to read the following recollection of his former student Jay Orear, presented during a meeting to celebrate the 2001 centenary of Fermi's birth: *"In my thesis I had to find the best 3-parameter fit to my data and the errors of those parameters in order to get the 3 phase shifts and their errors. Fermi showed me a simple analytic method. At the same time other physicists were using and publishing other cumbersome methods. Also Fermi taught me a general method, which he called Bayes Theorem, where one could easily derive the best-fit parameters and their errors as a special case of the maximum-likelihood method"*

Presently this recollection is included in the freely available Orear's book "Enrico Fermi, the master scientist" (`http://hdl.handle.net/1813/74`). So we can now learn that Fermi was teaching his students a maximum likelihood method *"derived from his Bayes Theorem"* and that *"the Bayes Theorem of Fermi"* - so Orear calls it - is a special case of Bayes Theorem, in which the priors are equally likely (and this assumption is explicitly stated!). Essentially, Fermi was teaching his young collaborators to use likelihood ratio to quantify how the data preferred one hypothesis among several possibilities, or to use the normalized likelihood to perform parametric inference (including the assumption of Gaussian approximation of the final pdf that simplifies the calculations).

Fermi was, among the other things, an extraordinary teacher, a gift witnessed by his absolute record in number of pupils winning the Nobel prize - up to about a dozen, depending on how one counts them. But in the case of probability based data analysis, it seems that his pupils didn't get fully the spirit of the reasoning and, when they remained orphans of their untimely dead scientific father, they were in an uneasy position between the words of

the teacher and the dominating statistical culture of those times. Bayes theorem, and especially his application to data analysis, appears in the Orear's book as one of the Fermi's working rules, of the kind of the 'Fermi golden rule' to calculate reaction probabilities. Therefore Orear reports of his ingenuous question to know *"how and when he learned this"* (how to derive maximum likelihood method from a more general tool). Orear *"expected him to answer R.A. Fisher or some textbook on mathematical statistics"*. *"Instead he said, 'perhaps it was Gauss' "*. And, according to his pupil, Fermi *"was embarrassed to admit that he had derived it all from his Bayes Theorem"*.

This last quote from the Orear's book gives an idea of the author's unease with that mysterious theorem and of his reverence for his teacher: *"It is my opinion that Fermi's statement of Bayesian Theorem is not the same as that of the professional mathematicians but that Fermi's version is nonetheless simple and powerful. Just as Fermi would invent much of physics independent of others, so would he invent mathematics"*.

Unfortunately, Fermi has written nothing on the subject. The other indirect source of information we have are the "Notes on statistics for physicists", written by Orear in 1958, where the author acknowledges that his *"first introduction to much of the material here was in a series of discussions with Enrico Fermi"* and others *"in the autumn 1953"* (Fermi died the following year). A revised copy of the notes is available on the web (`http://nedwww.ipac.caltech.edu/level5/Sept01/Orear/frames.html`).

When I saw the titles of the first two sections, "Direct probability" and "Inverse probability", I was hoping to find there a detailed account of the Fermi's Bayes Theorem. But I was immediately disappointed. Section 1 starts saying that *"books have been written on the 'definition' of probability"* and the author abstains from providing one, jumping to two properties of probability: statistical independence (not really explained) and the law of large numbers, put in a way that could be read as Bernoulli theorem as well as the frequentist definition of probability.

In Section 2, "Inverse probability", there is no mention to Bayes theorem, or to the Fermi's Bayes Theorem. Here we clearly see the experienced physicist tottering between the physics intuition, quite 'Bayesian', and the academic education on statistics, strictly frequentist (I have written years ago about this conflict and its harmful conse-

quences, see `http://xxx.lanl.gov/abs/physics/9811046`). Therefore Orear explains *"what the physicist usually means"* by a result reported in the form 'best value ± error': the physicist *"means the 'probability' of finding"* *"the true physical value of the parameter under question"* in the interval '[best value - error, best value + error]' is such and such percent. But then, the author immediately adds that *"the use of the word 'probability' in the previous sentence would shock the mathematician"*, because *"he would say that the probability"* that the quantity is in that interval *"is either 0 or 1"*. The section ends with a final acknowledgments of the conceptual difficulty and a statement of pragmatism: *"the kind of probability the physicist is talking about here we shall call inverse probability, in contrast to the direct probability used by the mathematicians. Most physicists use the same word, probability, for the two different concepts: direct probability and inverse probability. In the remainder of this report we will conform to the sloppy physics-usage of the word 'probability' "*.

Then, in the following sections he essentially presents a kind of hidden Bayesian approach to model comparison (only simple models) and parametric inference under the hypothesis of uniform prior, under which his guiding Fermi's Bayes Theorem held.

Historians and sociologists of science might be interested in understanding the impact that the Orear's notes have had in books for physicists written in the last forty-fifty years, and wonder how they would have been if the word 'Bayes' had been explicitly written in the notes.

Another question, which might be common to many readers at this point, is why Fermi associated Gauss' name to Bayes theorem. I am not familiar with all the original work of Gauss and a professional historian would be more appropriate. Anyway, I try to help with the little I know. In the derivation of the normal distribution (pp. 205-212 of his 1809 "Theoria motus corporum coelestium in sectionibus conicis solem ambientum" – I gave a short account of these pages in a book), Gauss develops a reasoning to invert the probability which is exactly Bayes theorem for hypotheses that are *a priori* equally likely[1] (the concepts of prior and posterior are well stated by Gauss), and, later, he extends the reasoning to the case of continuous variables. That is essentially what Fermi taught his collaborators. But Gauss never mentions Bayes, at least in the cited pages, and the use of the 'Bayesian' reasoning is different from what we usually do: we start

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[1]Something similar, also independently from Bayes, had done by Laplace in 1774 (see Stephen Stigler's 'The History of Statistics'). However Gauss does not mention Laplace for this result in his 1809 book (while, instead, he acknowledges him for the integral of the Gaussian!). Therefore the 'Fermi Bayes Theorem' could be, more properly, a kind of 'Laplace-Gauss Theorem'.

from likelihood and (often uniform) prior to get the posterior. Instead, Gauss got a general form of likelihood (his famous error distribution) from the assumptions of uniform prior, of the same error function for measurements and of maximum posterior at the arithmetic average of data points (plus some analytic property of the searched for function).

Then, why does Fermi mention Gauss for the name of the theorem and for the derivation of the maximum likelihood method from the theorem? Perhaps he had in mind another work of Gauss. Or it could be – I tend to believe more this second hypothesis – a typical Fermi unreliability in providing references, like in the following episode reported by Lincoln Wolfenstein in his contribution to the Orear's book: *"I remember the quantum mechanics course, where students would always ask, 'Well, could you tell us where we could find that in a book'' And Fermi said, grinning, 'It's in any quantum mechanics book!' He didn't know know any. They would say, 'well, name one!' 'Rojanski', he said,* 'it's in Rojanski'. *Well, it wasn't in Rojanski – it wasn't in any quantum mechanics book."*

I guess that, also in this case, most likely *it wasn't in Gauss*, though some seeds *are* in Gauss. In the pages that immediately follow his derivation of the normal distribution, Gauss shows that, using *his* error function, with the same function for all measurements, the posterior is maximized when the sum of the squares of residual is minimized. He recovered then the already known least square principle, that he claims to be his principle (*"principium nostrum"*, in Latin), though he acknowledges that Legendre had published a similar principle in 1806. Therefore, since Gauss used a flat prior, his 'Bayesian' derivation of the least square method is just a particular case of the maximum likelihood method. Fermi must have had this in mind, together with Bayes' name from modern literature and with many logical consequences that were not really in Gauss, when he replied young Orear, .