Bayes School 2013 at Stellenbosh

Problems based on lectures by Giulio d'Agostini

1. Scientists (of any field), checking *their* 'Standard Model' ('SM' – whatever it might be) against data by a χ^2, report a p-value of 0.001 .

   Which of the following statement(s) is/are correct?
   (It could be also ALL or NONE of them)

   (a) There is 99.9% probability that something new has been discovered.

   (b) There is only 0.1% probability that their SM is correct.

   (c) The probability they have observed a fluctuation is as low as 0.1%.

   (d) We can be 99.9% sure that the SM alone cannot describe the Real World.

   (e) The probability of the observed data given the SM has 0.1% probability.

2. A box contains a large number of black and white balls in equal proportions. A ball is extracted at random and, without watching its color, it is put into a (little) bag.

   Then, an extra <u>white</u> ball is added into the bag, the bag is shacked and a ball is extracted at random. This ball results to be white. How much should we be confident that the ball remaining in the box is also white?

3. A particle detector is placed into a beam of particles containing 90 % π ('pions') and 10 % μ ('muons') in order to 'tag' muons. The detector has a μ identification efficiency of 95 %, and a probability of identifying a π as a μ of 2 %. If a particle is identified as a μ, then a trigger is fired.

   (a) Calculate the probability that the trigger fires if a particle of the beam hits the detector.

   (b) Calculate the probability that a trigger is really fired by a μ particle. (Note: knowing the properties of muons and pions is irrelevant.)

   (c) Calculate the Signal to Noise ratio (S/N), meant as probability of muon over the probability of pion, given a trigger.

   (d) Calculate how much should have been the *Bayes Factor* in order to be 95% confident that a trigger would have been caused by a μ.

   (e) Assuming that we can arrive at a μ identification efficiency of *practically* 100% (could be 98%, 99% or more, the exact digits do not matter), calculate to which level we should reduce the π 'misinterpretation probability' (that originally was 2 %) in order to achieve such a *Bayes Factor*.

4. An <u>Italian</u> citizen is selected at <u>random</u> to undergo an AIDS test. The performance of clinical trial is not perfect, as customary. Let's assume the following *toy model*:

$$\begin{aligned} P(\text{Pos}\,|\,\text{HIV}) &= 100\% \\ P(\text{Pos}\,|\,\overline{\text{HIV}}) &= 0.2\% \\ P(\text{Neg}\,|\,\overline{\text{HIV}}) &= 99.8\%, \end{aligned}$$

where 'Pos' and 'Neg' are the possible results of the clinical trials (Positive/Negative), while 'HIV' and '$\overline{\text{HIV}}$' stand for "infected by HIV" and "not infected by HIV". Let us also assume that the number of infected people in Italy is estimated to about 100000 in a population of about 60 millions.
The result says: <u>Positive</u>.

(a) Calculate the probability that such a person is infected (it is recommended to use odds and Bayes Factor).

(b) Immediately after, the same person undergoes to a second, <u>independent</u> trial with exactly the same performances of the first one, and the result is again <u>Positive</u>. Calculate the probability that the person is infected, based on the two tests.
(Again, using odds and Bayes Factors speeds up the calculations.)

(c) Imagine, instead, that the two clinical trials have, for some biochemical reasons, some degree of correlation in the case a person is not infected, such that there is 10% probability that the second trials gives Positive if the first one has given Positive, that is

$$P(\text{Pos}^{(2)}\,|\,\text{Pos}^{(1)},\overline{\text{HIV}}) = 0.1$$

where $\text{Pos}^{(1)}$ and $\text{Pos}^{(2)}$ are the results of the two trials.
Calculate how this information changes the probability we have to assign to the person to be infected in the light of the two results.
[I.e. → calculate $P(\text{HIV}\,|\,\text{Pos}^{(1)},\text{Pos}^{(2)})$ and, again, using Bayes Factors helps.]

5. Imagine we are interested in measuring the efficiency of a new detector having a small area. Using suitable other detectors with stringent conditions on the trigger we are practically sure that 100 particles of a given type hit the detector under study. As a result, our detector has produced 100 times an electric signal over threshold. What can we say about the detector efficiency ϵ under the hypothesis that all possible values of ϵ between 0 and 1 were *a priori* equally likely? In particular, give

(a) the probability density function $f(\epsilon\,|\,\text{data})$.

(b) the most probable value of ϵ (i.e. the 'mode' ϵ_m);

(c) the expected value of ϵ (i.e. $\mathrm{E}[\epsilon]$).

Then

(d) answer again the last two questions making use of the properties of the Beta pdf, calculating also the *standard uncertainty* concerning ϵ

6. Our prior knowledge about the parameter p of a binomial distribution is such that we believe it should be about 0.75, with a standard uncertainty of 0.10. Than we make an 'experiment' under stable conditions (we believe that p remains constant in all trials) resulting in 50 successes and 50 failures. Making use of the beta conjugated pdf calculate how the data update our knowledge about p in terms of expected value and standard uncertainty.
[**Hint**: during the calculations make use of the approximation $(r+s) \gg 1$, such that $r+s+1 \approx r+s$.]

7. A physicist measures two homogeneous quantities with the same experimental device, getting, in an arbitrary unit 'u'

$$\begin{aligned} \mu_1 &= 1.05 \pm 0.04\,\mathrm{u} \\ \mu_2 &= 1.23 \pm 0.04\,\mathrm{u}\ , \end{aligned}$$

where the reported standard uncertainties take only account 'statistical errors'. But we know that the experiment could be affected by an overall *additive* ('offset') *systematic error* which is believed to be 0, with a standard uncertainty of 0.10 u, having calibrated at best all detectors and analysis procedures.

Taking into account this extra piece of information, evaluate

(a) the global uncertainty on μ_1 and μ_2;

(b) the correlation coefficient between the two results;

(c) expected value and uncertainty about $\Delta\mu = \mu_2 - \mu_1$;

(d) expected value and uncertainty about $M = \mu_1 + \mu_2$.